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Abstract. Our primary objective in this article is to compare the spectral densities of numerous cyclostationary processes. By using the limiting distributions of the periodogram and the discrete Fourier transform, a novel approach is introduced to compare the spectral densities of independent cyclostationary processes. Also, the ability of the introduced approach is studied by employing simulated and real datasets.

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1. Introduction

Comparison of several processes is a main subject in economics, physics, chemistry, signal processing and materials. Really, the researchers try to compare the stochastic mechanism of some observed datasets from different time series. The classification, confrontation and clustering of two or numerous time series processes were investigated in different time-domain and frequency-domain methods by several scientists [see e.g. De Souza and Thomson [1], Coates and Diggle [2], Potscher and Reschenhofer [3], Diggle and Fisher [4], Dargahi- Noubary [5], Diggle and al Wasel [6], Kakizawa et al. [7], Timmer et al. [8], Maharaj [9, 10], Caiado et al. [11], Eichler [12], Fokianos and Savvides [13], Caiado et al. [14], Dette and Paparoditis [15], Dette et al. [16], Dette and Hildebrandt [17], Jentsch [18], Jentsch and Pauly [19], Salcedo et al. [20], Jentsch and Pauly [21], Mahmoudi et al. [22], Triacca [23]. Nearly all these approaches can be utilized to the stationary processes or the non-stationary processes (which are transformable to stationary processes by using differencing). Nevertheless this approach will not apply to numerous processes. Maharaj [24, 25] investigated the problem for non-stationary time series that can be transformed to stationary processes.

Cyclostationary (CS) processes that are presented by Gladyshev [26] are nicely applied to depict rhythmic processes. The CS processes (may be also called periodically correlated in statistics) are a big time series group with cyclic mean and auto-covariance functions. These periodicities can not be disconnected by transformation consist of differencing. These processes can be fitted on real datasets of sciences, like economics, physics, chemistry, signal processing and materials (Gardner et al. [27]). Hurd and Miamee [28], Napolitano [29] and Chaari et al. [30] are suitable references about the applications as well as theories of CS processes. Hurd and Gerr [31] considered the detection of periodicity by two graphical approaches and using the coherency and incoherency. Broszkiewicz-Suwaj [32] applied the bootstrap methodology and constructed a measure of fitness statistic (called MoF, in short) to detect the periodicity. Nematollahi et al. [33] used periodogram asymptotic distribution to establish a goodness of fit test for CS time series. Mahmoudi and Maleki [34] studied the detection of periodicity using the estimating of the spectral support of a given dataset.

In present research, by using the limiting distributions of the periodogram and the discrete Fourier transform of CS processes, a new method is introduced to investigate the equality of several CS processes. Also, the ability of the introduced approach is studied by using simulation study and real dataset. The remainder of paper is structured as follows. Section 2 is devoted to the notations and preliminaries. The methodology to compare the CS models will be given in Section 3. Section 4 will report the results of simulation study and real world dataset to investigate the ability of the introduced approach.

2. Preliminaries

Definition 1: A time series $\{X_t, t \in \mathbb{Z}\}$, is CS with period T (CS-T), if

$$m(t) := E(X_t) = m(t+T),$$

and

$$R(s,t) := Cov(X_s, X_t) = E\left[(X_s - m(s))\overline{(X_t - m(t))}\right] = R(s+T, t+T),$$

for $s, t \in \mathbb{Z}$.

A time CS-T can be expressed as the following spectral representation on $[0, 2\pi)$,

$$X_t = \int_0^{2\pi} e^{itx}\Psi(dx), \quad t \in \mathbb{Z},$$

such that Ψ is a random measure with following property on $[0, 2\pi)$:

$$E\left(\Psi(d\lambda)\overline{\Psi(d\lambda')}\right) = 0, \lambda - \lambda' \neq 2\pi k, k = -T+1, \dots, T-1, k \neq 0.$$

We can define the spectral distribution of Ψ, by

$$\boldsymbol{F}(d\lambda) = \left[F_{k-j}\left(d\lambda + \frac{2\pi j}{T}\right)\right]_{j,k=0,\dots,T-1}, \quad \lambda \in \left[0, \frac{2\pi}{T}\right),$$

where

$$F_k(d\lambda) = E\left(\Psi(d\lambda)\overline{\Psi\left(d\lambda + \frac{2\pi k}{T}\right)}\right) = F\left(d\lambda, d\lambda + \frac{2\pi k}{T}\right), k = -T+1, \dots, T-1.$$

Also the spectral density of Ψ, $\boldsymbol{f} = [f_{jk}]_{j,k=0,\dots,T-1}$, is defined by

$$\boldsymbol{f}(\lambda) = \frac{d\boldsymbol{F}}{d\lambda} = \left[f_{k-j}\left(\lambda + \frac{2\pi j}{T}\right)\right]_{j,k=0,\dots,T-1}, \quad \lambda \in \left[0, \frac{2\pi}{T}\right),$$

such that the density f_k is corresponding to the F_k.

3. Methodology

Let $\{X_t^{(1)}, t = 1, \dots, n_1\}$, $\{X_t^{(2)}, t = 1, \dots, n_2\}$, ..., $\{X_t^{(m)}, t = 1, \dots, n_m\}$, are observations from *m* independent CS processes with period *T*.

In many fields the researchers are interested to compare the spectral densities of the time series $X_t^{(1)}$, …, and $X_t^{(m)}$. In other words, they want to test following hypothesis:

$$H_0: \boldsymbol{f}_1 = \boldsymbol{f}_2 = \cdots = \boldsymbol{f}_m,$$

such that $\boldsymbol{f}_1, \dots,$ and $\boldsymbol{f}_m$ are respectively the spectral density matrices of the time series $X_t^{(1)}$, …, and $X_t^{(m)}$.

Under the rejection of H_0, we conclude that at least two time series have different spectral densities, and if H_0 is accepted, consequently there is not meaningful different between the spectral densities of the *m* processes and the stochastic mechanisms of *m* processes are the same.

Assume $X_0, \dots, X_{n-1}$ are the observations from X_t. The discrete Fourier transform of these observations is defined by

$$d_X(\lambda) = n^{-1/2} \sum_{t=0}^{n-1} X_t e^{it\lambda} \;, \lambda \in [0, 2\pi).$$

The periodogram of CS time series was defined by Soltani and Azimmohseni [35] as

$$\boldsymbol{I}_X^T(\lambda) = \boldsymbol{d}_X^T(\lambda) \boldsymbol{d}_X^{T^*}(\lambda),$$

where

$$\boldsymbol{d}_X^T(\lambda) = \left(d_X(g_1(\lambda)), d_X(g_2(\lambda)) \dots, d_X(g_T(\lambda))\right)' \;, \lambda \in \left[0, \frac{2\pi}{T}\right),$$

such that

$$g_k(x) = x + \frac{2\pi(k-1)}{T},$$

for $k = 1, \dots, T$.

Lemma 3.1: (**Soltani and Azimmohseni [35])**

For a CS-T processes X_t, assume $\boldsymbol{f}(\lambda),\ \lambda \in [0,2\pi)$ is continuous. If $\lambda_1 < \cdots < \lambda_J$ are frequencies in $\left[0, \frac{2\pi}{T}\right)$, then

(i) $\hat{\boldsymbol{f}}(\lambda) := \frac{\boldsymbol{I}_X^T(\lambda)}{2\pi}$ asymptotically estimated $\boldsymbol{f}(\lambda),\ \lambda \in \left[0, \frac{2\pi}{T}\right)$.

(ii) $\boldsymbol{d}_X^T(\lambda_j), j = 1, \dots, J,$ have asymptotical complex normal distributions

$$N_T^c\left(0, 2\pi\boldsymbol{f}(\lambda_j)\right).$$

(iii) $\boldsymbol{d}_X^T(\lambda_j), j = 1, \dots, J,$ are asymptotically independent.

(iv) $\boldsymbol{I}_X^T(\lambda_j), j = 1, \dots, J,$ have asymptotical complex Wishart distributions

$$W_T^c\left(\mathbf{1},\ 2\pi\boldsymbol{f}(\lambda_j)\right).$$

(v) $\boldsymbol{I}_X^T(\lambda_j), j = 1, \dots, J,$ are asymptotically independent.

For $j = 1, \dots, J$, assume

$$Y_j^{(k)} = Re\left(\boldsymbol{d}_{X^{(k)}}^T(\lambda_j)\right), k = 1,2, \dots, m,$$

and

$$Z_j^{(k)} = Im\left(\boldsymbol{d}_{X^{(k)}}^T(\lambda_j)\right), k = 1,2, \dots, m,$$

where $\boldsymbol{d}_{X^{(k)}}^T(\lambda_j)$ is $\boldsymbol{d}_X^T(\lambda_j)$ for the population k^{th}.

Corollary 3.1: Let

$$W_j^{(k)} = \left(Y_j^{(k)}, Z_j^{(k)}\right)', k = 1,2, \dots, m, \qquad j = 1, \dots, J.$$

Then for $j = 1, \dots, J$,

(i) $W_j^{(k)}, k = 1,2, \dots, m,$ are asymptotically independent.

(ii) The asymptotic distribution of $W_j^{(k)}, k = 1,2, \dots, m,$ is $N_{2T}\left(0, \boldsymbol{\Sigma}_j^{(k)}\right)$, where

$$\mathbf{\Sigma}_j^{(k)} = \begin{bmatrix} \boldsymbol{V}_{Y_jY_j}^{(k)} & \boldsymbol{V}_{Y_jZ_j}^{(k)} \\ \boldsymbol{V}_{Z_jY_j}^{(k)} & \boldsymbol{V}_{Z_jZ_j}^{(k)} \end{bmatrix}, \boldsymbol{V}_{AB} = COV(A,B).$$

Proof: This is a straight result of previous lemma.

Consequently,

$$U^{(k)} = \sum_{j=1}^{J} W_j^{(k)}, k = 1,2,\dots,m,$$

is asymptotically $N_{2T}\left(0, \mathbf{\Sigma}^{(k)}\right)$, such that

$$\mathbf{\Sigma}^{(k)} = \mathbf{\Sigma}_1^{(k)} + \cdots + \mathbf{\Sigma}_J^{(k)}.$$

3.1. Testing problem

As discussed, in practice the researchers are interested to test

$$H_0: \boldsymbol{f_1} = \boldsymbol{f_2} = \cdots = \boldsymbol{f_m}.$$

This hypothesis is equivalent to

$$H_0: \mathbf{\Sigma}^{(1)} = \mathbf{\Sigma}^{(2)} = \cdots = \mathbf{\Sigma}^{(m)},$$

As a consequence, the asymptotic distribution of

$$U = U^{(1)} + U^{(2)} + \cdots + U^{(m)}$$

is $N_{2T}(0, \mathbf{\Sigma})$, where

$$\mathbf{\Sigma} = \mathbf{\Sigma}^{(1)} + \mathbf{\Sigma}^{(2)} + \cdots + \mathbf{\Sigma}^{(m)}.$$

Therefore the asymptotic distribution of the statistic

$$\chi^2 = (U)'(\mathbf{\Sigma})^{-1}(U),$$

is $\chi^2(2T)$.

Therefore, the asymptotic distribution is applied to establish test of hypothesis about H_0.

The statistic χ^2 is related to unknown parameter $\mathbf{\Sigma}$. Therefore first this parameter should be estimate. Let

$$\boldsymbol{S} = \frac{(N_1 - 1)\mathbf{S}^{(1)} + (N_2 - 1)\mathbf{S}^{(2)} + \cdots + (N_m - 1)\mathbf{S}^{(m)}}{N_1 + N_2 + \cdots + N_m - m},$$

as the sample pooled covariance matrix, where

$$\mathbf{S}^{(k)} = \mathbf{S}_1^{(k)} + \cdots + \mathbf{S}_J^{(k)},\ \mathbf{S}_j^{(k)} = \begin{bmatrix} \widehat{\boldsymbol{V}}_{Y_jY_j}^{(k)} & \widehat{\boldsymbol{V}}_{Y_jZ_j}^{(k)} \\ \widehat{\boldsymbol{V}}_{Z_jY_j}^{(k)} & \widehat{\boldsymbol{V}}_{Z_jZ_j}^{(k)} \end{bmatrix}, \text{ and } \widehat{\boldsymbol{V}}_{AB} = \widehat{COV}(A,B).$$

If H_0: $\mathbf{\Sigma}^{(1)} = \mathbf{\Sigma}^{(2)} = \cdots = \mathbf{\Sigma}^{(m)}$, be true, then $\boldsymbol{S}$ can be consistently estimated $\mathbf{\Sigma}$, and consequently as a result of Weak Law of Large Numbers, the asymptotic distribution of

$$\chi^{2^*} = (U)'(\mathbf{S})^{-1}(U),$$

is $\chi^2(2T)$. Therefore the hypothesis H_0 is rejected if $\chi^{2^*} > \chi^2_{1-\alpha}(2T)$, where α is size of test.

Remark 1: In real problem, we need more samples from $\boldsymbol{d}_{X^{(k)}}^T$ (N_k samples for population $k^{th}, k = 1,2, \dots, m$). The bootstrap estimation methods can be applied to reach this aim.

In this work, the moving block bootstrap methodology (MBB, in short) [36] will be used.

4. Simulation Study

This section is devoted to study the ability of introduced technique for simulated datasets. The steps of simulation procedure are as following:

(i) For the first, the second and the third time series, respectively, simulate a sample of size n_1, n_2 and n_3.

(ii) Calculate $\boldsymbol{d}_X^T(\lambda_j), j = 1, \dots, J$, separately, for simulated samples.

(iii) Previous steps are repeated 1000 times to provide 1000 samples for $\boldsymbol{d}_X^T(\lambda_j), j = 1, \dots, J$.

(iv) Calculate the value of χ^{2^*} and then compare it with $\chi^2_{1-\alpha}(2T)$.

(v) Previous steps are repeated one thousand times to estimate the level and the power of the test that can be respectively computed by

$$\hat{\alpha} = \frac{\text{The number of runs for which the value of } \chi^{2^*} \text{ is more than } \chi^2_{1-\alpha}(2T), \text{under } \mathrm{H}_1}{1000},$$

$$\hat{\pi} = \frac{\text{The number of runs for which the value of } \chi^{2^*} \text{ is more than } \chi^2_{1-\alpha}(2T), \text{under } \mathrm{H}_1}{1000}.$$

Example 1: Suppose the time series

$$X_t^{(i)} = \phi_t^{(i)} X_{t-1}^{(i)} + Z_t^{(i)}, \qquad \left\{Z_t^{(i)}\right\} \sim IIDN(0,1), \;\; i = 1,2,3,$$

such that

$\phi_t^{(i)} = 0.6 + \phi^{(i)} \cos\left(\frac{2\pi t}{T}\right), \phi^{(1)} = 0.5, \phi^{(2)} = 0.1, 0.5$ and $\phi^{(3)} = 0.5,0.7$.

Example 2: Suppose the time series

$$X_t = Z_t^{(i)} + \theta_t^{(i)} Z_{t-1}^{(i)}, \qquad \left\{Z_t^{(i)}\right\} \sim IIDN(0,1), \;\; i = 1,2,3,$$

such that

$\theta_t^{(i)} = 1 + \theta^{(i)} \cos\left(\frac{2\pi t}{T}\right), \theta^{(1)} = 0.5, \theta^{(2)} = 0.3, 0.5$ and $\theta^{(3)} = 0.5,0.9$.

Example 3: Suppose the time series

$$X_t^{(i)} - \phi_t^{(i)} X_{t-1}^{(i)} = Z_t^{(i)} + \theta_t^{(i)} Z_{t-1}^{(i)}, \qquad \left\{Z_t^{(i)}\right\} \sim IIDN(0,1), \;\; i = 1,2,3,$$

such that

$\phi_t^{(i)} = 0.6 + \phi^{(i)} \cos(2\pi t/T), \;\; \theta_t^{(i)} = 1 + \theta^{(i)} \cos\left(\frac{2\pi t}{T}\right), \phi^{(1)} = 0.2, \theta^{(1)} = -0.5, \;\; \phi^{(2)} = 0.2,$ $\theta^{(2)} = -0.1, , -0.5, \phi^{(3)} = 0.2,0.4$ and $\theta^{(3)} = -0.5$.

Example 4: Suppose the time series

$$X_t^{(i)} = (1 + m^{(i)} \cos(2\pi t/T)) Z_t^{(i)}, \qquad \left\{Z_t^{(i)}\right\} \sim IIDN(0,1), \;\; i = 1,2,3,$$

where

$m^{(1)} = 0.5, m^{(2)} = 0.1, 0.5$ and $m^{(3)} = 0.5, 0.7$.

The estimated $\hat{\alpha}$ with *T=2, 3, 4,* and *5*, for Examples 1-4, respectively, are summarized in third rows of Tables 1 to 4, respectively. Since these values are very adjacent to the $\alpha = 0.05$, particularly when $(\mathrm{n}_1, \mathrm{n}_2, \mathrm{n}_3)$ grows, then the proposed approach controlled the type I error ($\alpha = 0.05$). Other rows of Tables 1 to 4 are correspond to the values of $\hat{\pi}$. These values verify the excellent ability of the introduced technique to discriminate between H_0 and H_1. Also the Q–Q plots for the values of the test statistic χ^{2^*} against the $\chi^2(2T)$ distribution have been presented in Figure 1. As can be seen the points are close to strain line and consequently the test statistic χ^{2^*} is asymptotically $\chi^2(2T)$. Therefore presented approach performs well in simulation.

5. Real Data

This section is devoted to study the proposed approach in real world problem. The considered real dataset is the logarithms of the Real Gross Domestic Product in Germany [37]. We divide whole dataset in four separate sections; section one: from spring 1960 to winter 1967, section two: from spring 1968 to winter 1975, section three: from spring 1976 to winter 1984, and section four: from spring 1985 to winter 1990. Figure 2 is the CSS plots [34] of these sections. These plots, determine a CS-4 time series for all sections that verify the time series given by [37]. Therefore all of sections follow from a CS-4 model. Now the given approach will be used to test the hypothesis $H_0: \mathbf{\Sigma}^{(1)} = \mathbf{\Sigma}^{(2)} = \mathbf{\Sigma}^{(3)} = \mathbf{\Sigma}^{(4)}$ (or equivalently, $H_0: \boldsymbol{f_1} = \boldsymbol{f_2} = \boldsymbol{f_3} = \boldsymbol{f_4}$). Table 5 summarized the results. Since the p-value is more than 0.05 (p=0.363), consequently we accept all of sections have similar spectral densities. For future studies investigation of further application domains, e.g., [38-59] using the proposed method is suggested.

6. Conclusion

Comparison of several time series is a main subject in economics, physics, chemistry, signal processing and materials. Really, the researchers try to compare the stochastic mechanism of some observed time series. The classification, confrontation and clustering of two or numerous time series processes were investigated in different time-domain and frequency-domain methods by several scientists. Nearly all these approaches can be utilized to the stationary processes or the non-stationary processes (which are transformable to stationary processes by using differencing). Nevertheless this approach will not apply to numerous processes. This paper was devoted to compare the spectral densities of numerous uncorrelated cyclostationary processes. By using the limiting distributions of the periodogram and the discrete Fourier transform, a novel approach was introduced to compare the spectral densities of independent cyclostationary processes. Also, the ability of the introduced approach was studied by using simulation and real examples. The presented technique performed well in simulation. This proposed method is also controlled the type I error and verified the excellent ability to discriminate between H_0 and H_1.

Conflict of Interest

The authors declare that they have no conflict of interest.

Table 1: The level and power of the proposed technique for the first example

		(n_1, n_2, n_3)			
$\boldsymbol{\phi^{(2)}}$	$\boldsymbol{\phi^{(3)}}$	(100, 50,75)	(150, 75,100)	(200, 150,100)	(500, 250,300)
0.1	**0.5**	0.832	0.912	0.995	0.999
0.1	**0.7**	0.864	0.964	0.997	1.000
0.5	**0.5**	0.051	0.050	0.049	0.049
0.5	**0.7**	0.812	0.952	0.997	1.000

Table 2: The level and power of the proposed technique for the second example

		(n_1, n_2, n_3)			
$\boldsymbol{\theta^{(2)}}$	$\boldsymbol{\theta^{(3)}}$	(100, 50,75)	(150, 75,100)	(200, 150,100)	(500, 250,300)
0.3	**0.5**	0.802	0.943	0.995	1.000
0.3	**0.9**	0.865	0.965	0.997	1.000
0.5	**0.5**	0.051	0.050	0.050	0.050
0.5	**0.9**	0.843	0.925	0.998	1.000

Table 3: The level and power of the proposed technique for the third example

		(n_1, n_2, n_3)			
$\boldsymbol{\theta^{(2)}}$	$\boldsymbol{\phi^{(3)}}$	$(100, 50,75)$	$(150, 75,100)$	$(200, 150,100)$	$(500, 250,300)$
-0.1	**0.2**	0.811	0.923	0.992	1.000
-0.1	**0.4**	0.821	0.941	0.989	1.000
-0.5	**0.2**	0.051	0.050	0.050	0.049
-0.5	**0.4**	0.834	0.954	0.991	0.999

Table 4: The level and power of the proposed technique for the fourth example

		(n_1, n_2, n_3)			
$\boldsymbol{m^{(2)}}$	$\boldsymbol{m^{(3)}}$	$(100, 50,75)$	$(150, 75,100)$	$(200, 150,100)$	$(500, 250,300)$
0.1	**0.5**	0.843	0.934	0.999	1.000
0.1	**0.7**	0.876	0.965	0.998	1.000
0.5	**0.5**	0.050	0.050	0.049	0.049
0.5	**0.7**	0.843	0.954	0.998	1.000

Table 5: The results of the proposed approach to compare the different sections of the logarithms of the Real Gross Domestic Product in Germany

Test Statistic	**P-Value**
$\chi^{2^*} = 8.759$	0.363

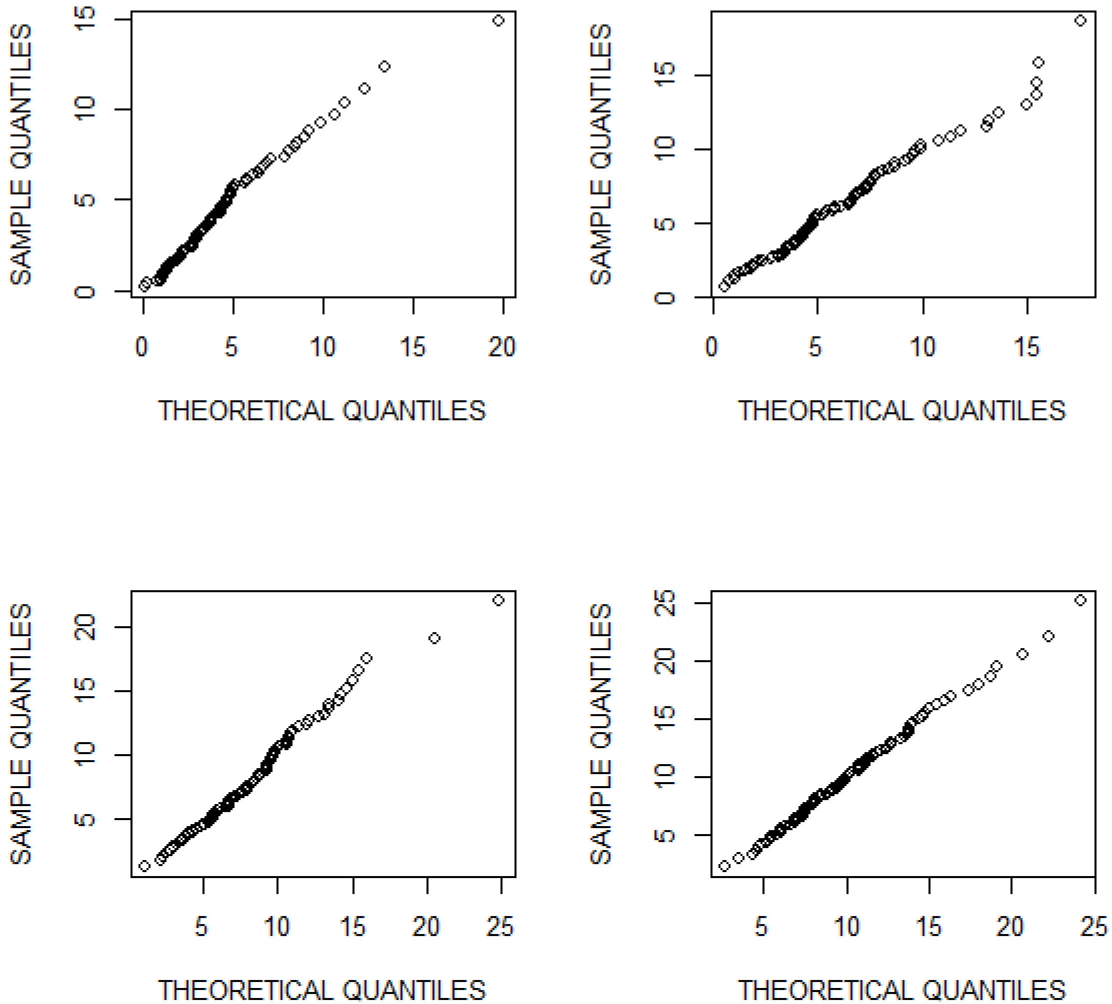

Figure 1: Q–Q plots for the values of the test statistic χ^{2^*} against the $\chi^2(2T)$ distribution

First row, First column is corresponded to *Example 1 with sample sizes* $(n_1, n_2, n_3) = (100{,}50{,}75)$.

First row, Second column is corresponded to *Example 2 with sample sizes* $(n_1, n_2, n_3) = (150{,}75{,}100)$.

Second row, First column is corresponded to *Example 3 with sample sizes* $(n_1, n_2, n_3) = (200{,}150{,}100)$.

Second row, Second column is corresponded *to Example 4 with sample sizes* $(n_1, n_2, n_3) = (5002{,}50{,}300)$.

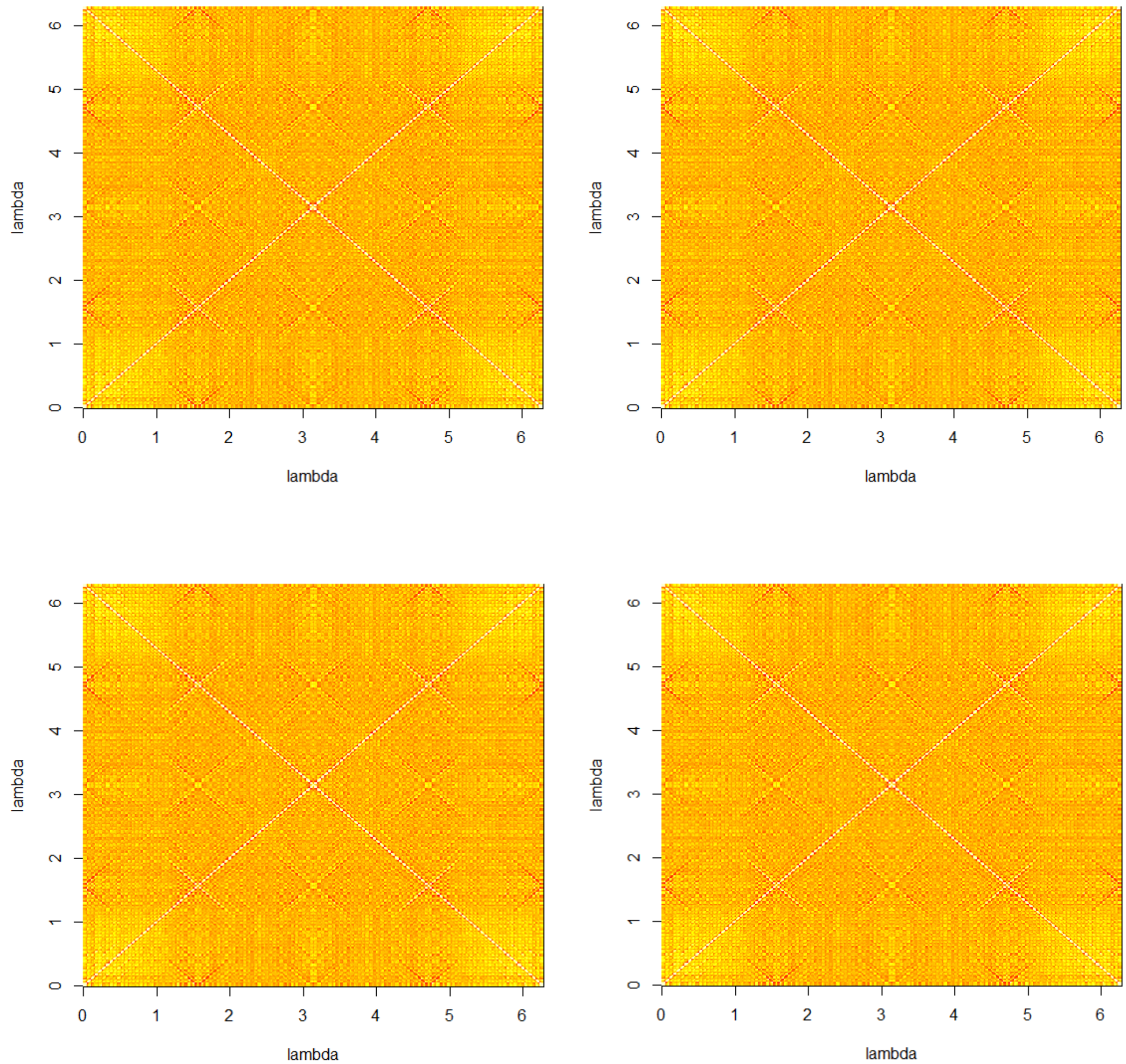

Figure 2: CSS plot for different sections (Above: Left: Section 1, Right: Section 2; Below: Left: Section 3, Right: Section 4)

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