Riemann Hypothesis on Superabundant Numbers

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To my mother

Abstract. The Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. It is considered by many to be the most important unsolved problem in pure mathematics. Let $\Psi(n) = n \cdot \prod_{q \mid n} \left(1 + \frac{1}{q}\right)$ denote the Dedekind Ψ function where $q \mid n$ means the prime q divides n. Define, for $n \geq 3$; the ratio $R(n) = \frac{\Psi(n)}{n \cdot \log \log n}$ where log is the natural logarithm. Let $\sigma(n)$ denote the sum-of-divisors function $\sigma(n) = \sum_{d \mid n} d$. We require the properties of superabundant numbers, that is to say left to right maxima of $n \mapsto \frac{\sigma(n)}{n}$. There are several statements equivalent to the Riemann hypothesis. If for each large enough superabundant number n, there exists another superabundant $n' > n$ such that $R(n') \leq R(n)$, then the Riemann hypothesis is true. In this note, using this criterion on superabundant numbers, we prove that the Riemann hypothesis is true.

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1. Introduction

The Riemann hypothesis was proposed by Bernhard Riemann (1859) [3]. The Riemann hypothesis belongs to the Hilbert's eighth problem on Hilbert's list of twenty-three unsolved problems [3]. This is one of the Clay Mathematics Institute's Millennium Prize Problems [3]. In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{q \leq x} \log q$$

with the sum extending over all prime numbers q that are less than or equal to x, where log is the natural logarithm.

Proposition 1.1. *We have* [12, pp. 1]*:*

$$\theta(x) \sim x \quad as \quad (x \to \infty).$$

The following property is based on natural logarithms:

Proposition 1.2. *For* $x > -1$ [8, pp. 1]*:*

$$\frac{x}{x+1} \leq \log(1+x).$$

Leonhard Euler studied the following value of the Riemann zeta function (1734) [2].

Proposition 1.3. *We define* [2, (1) pp. 1070]*:*

$$\zeta(2) = \prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1} = \frac{\pi^2}{6},$$

where q_k is the kth prime number. By definition, we have

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

where n denotes a natural number. Leonhard Euler proved in his solution to the Basel problem that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1} = \frac{\pi^2}{6},$$

where $\pi \approx 3.14159$ is a well-known constant linked to several areas in mathematics such as number theory, geometry, etc.

Proposition 1.4. *For $x \geq 3$ we have* [5, Lemma 6.4 pp. 370]*:*

$$\left(\prod_{q>x} \frac{q^2}{q^2 - 1}\right) \leq \exp\left(\frac{2}{x}\right),$$

where $\exp(k)$ is the exponential function with value e^k and exponent k. Indeed, Choie and her colleagues proved that for $x \geq 3$ and $t \geq 2$,

$$\log(R_t(x)) \leq \frac{t \cdot x^{1-t}}{t-1},$$

where $R_t(x)$ is given as

$$R_t(x) = \prod_{q>x} (1 - q^{-t})^{-1} = \prod_{q>x} \frac{q^t}{q^t - 1}.$$

Therefore, this Proposition is a particular case of their result applied to the specific value of $t = 2$.

The number $\gamma \approx 0.57721$ is the Euler-Mascheroni constant which is defined as

$$\gamma = \lim_{n\to\infty} \left(-\log n + \sum_{k=1}^{n} \frac{1}{k} \right)$$
$$= \int_{1}^{\infty} \left(-\frac{1}{x} + \frac{1}{\lfloor x \rfloor} \right) dx.$$

Here, $\lfloor \ldots \rfloor$ represents the floor function. As usual $\sigma(n)$ is the sum-of-divisors function of n

$$\sum_{d|n} d,$$

where $d \mid n$ means the integer d divides n. Define $I(n)$ as $\frac{\sigma(n)}{n}$ to be the abundancy index function. In 1997, Ramanujan's old notes were published where he defined the generalized highly composite numbers, which include the superabundant and colossally abundant numbers [11]. Superabundant numbers were also studied by Leonidas Alaoglu and Paul Erdős (1944) [1]. Let $q_1 = 2, q_2 = 3, \ldots, q_k$ denote the first k consecutive primes, then an integer of the form $\prod_{i=1}^{k} q_i^{a_i}$ with $a_1 \geq a_2 \geq \ldots \geq a_k \geq 1$ is called a Hardy-Ramanujan integer [5, pp. 367]. A natural number n is called superabundant precisely when, for all natural numbers $m < n$

$$I(m) < I(n).$$

We know the following properties of the superabundant numbers:

Proposition 1.5. *If n is superabundant, then n is a Hardy-Ramanujan integer* [1, Theorem 1 pp. 450].

Proposition 1.6. *Let n be a superabundant number such that q is the largest prime factor of n. Then* [1, Theorem 7 pp. 454]*:*

$$q \sim \log n \quad as \quad (n \to \infty).$$

In number theory, the *p-adic* order of an integer n is the exponent of the highest power of the prime number p that divides n. It is denoted $\nu_p(n)$. Equivalently, $\nu_p(n)$ is the exponent to which p appears in the prime factorization of n.

Proposition 1.7. *Let n be a superabundant number such that p is the largest prime factor of n and a prime number $2 \leq q \leq p$. Then* [9, Lemma 14 pp. 8]*:*

$$\left\lfloor \frac{\log p}{\log q} \right\rfloor \leq \nu_q(n).$$

In number theory, $\Psi(n) = n \cdot \prod_{q|n} \left(1 + \frac{1}{q}\right)$ is called the Dedekind Ψ function, where $q \mid n$ means the prime q divides n. A natural number N_k is called a primorial number of order k precisely when,

$$N_k = \prod_{i=1}^{k} q_i.$$

We define $R(n) = \frac{\Psi(n)}{n \cdot \log \log n}$ for $n \geq 3$.

Proposition 1.8. *Unconditionally on Riemann hypothesis, we know that* [13, Proposition 3. pp. 3]*:*

$$\lim_{k \to \infty} R(N_k) = \frac{e^{\gamma}}{\zeta(2)}.$$

Definition 1.9. If n is a superabundant number, then we say that $\mathsf{Dedekind}(n)$ holds provided that

$$\prod_{q \mid n} \left(1 + \frac{1}{q}\right) \geq \frac{e^{\gamma}}{\zeta(2)} \cdot \log \log n.$$

The well-known asymptotic notation Ω was introduced by Godfrey Harold Hardy and John Edensor Littlewood [6]. In 1916, they also introduced the two symbols Ω_R and Ω_L defined as [7]:

$$f(x) = \Omega_R(g(x)) \text{ as } x \to \infty \text{ if } \limsup_{x \to \infty} \frac{f(x)}{g(x)} > 0;$$
$$f(x) = \Omega_L(g(x)) \text{ as } x \to \infty \text{ if } \liminf_{x \to \infty} \frac{f(x)}{g(x)} < 0.$$

After that, many mathematicians started using these notations in their works. From the last century, these notations Ω_R and Ω_L changed as Ω_+ and Ω_-, respectively. There is another notation: $f(x) = \Omega_{\pm}(g(x))$ (meaning that $f(x) = \Omega_+(g(x))$ and $f(x) = \Omega_-(g(x))$ are both satisfied). Nowadays, the notation $f(x) = \Omega_+(g(x))$ has survived and it is still used in analytic number theory as:

$$f(x) = \Omega_+(g(x)) \text{ if } \exists k > 0 \, \forall x_0 \, \exists x > x_0 \colon f(x) \geq k \cdot g(x)$$

which has the same meaning to the Hardy and Littlewood older notation. For $x \geq 2$, the function f was introduced by Nicolas in his seminal paper as [10, Theorem 3 pp. 376], [4, (5.5) pp. 111]:

$$f(x) = e^{\gamma} \cdot \log \theta(x) \cdot \prod_{q \leq x} \left(1 - \frac{1}{q}\right).$$

Finally, we have the Nicolas Theorem:

Proposition 1.10. *If the Riemann hypothesis is false then there exists a real b with $0 < b < \frac{1}{2}$ such that, as $x \to \infty$* [10, Theorem 3 (c) pp. 376], [4, Theorem 5.29 pp. 131]*:*

$$\log f(x) = \Omega_{\pm}(x^{-b}).$$

Putting all together yields a proof for the Riemann hypothesis.

2. Central Lemma

Several analogues of the Riemann hypothesis have already been proved. Many authors expect (or at least hope) that it is true. Nevertheless, there exist some implications in case of the Riemann hypothesis could be false. The following is a key Lemma.

Lemma 2.1. *If the Riemann hypothesis is false, then there exist infinitely many superabundant numbers n such that* $\mathsf{Dedekind}(n)$ *fails (i.e.* $\mathsf{Dedekind}(n)$ *does not hold).*

Proof. The function g is defined as:

$$g(x) = \frac{e^{\gamma}}{\zeta(2)} \cdot (\log\log x) \cdot \prod_{q \mid x} \left(1 + \frac{1}{q}\right)^{-1}.$$

We can see that whenever there exists some superabundant number n' such that $g(n') > 1$ or equivalent $\log g(n') > 0$, then we obtain that $\mathsf{Dedekind}(n')$ fails as a direct consequence. We can prove the following bound:

$$\log g(n) \geq \log f(q_k) - \frac{2}{q_k}$$

where q_k is the largest prime factor of the superabundant number n. Certainly, we know that

$$(\log\log n) \geq \log \theta(q_k)$$

by Proposition 1.5. Moreover, we have

$$\log\left(\prod_{q \mid n} \left(1 + \frac{1}{q}\right)^{-1}\right) - \log\zeta(2) \geq \log\left(\prod_{q \leq q_k} \left(1 - \frac{1}{q}\right)\right) - \frac{2}{q_k}.$$

This is because of the Propositions 1.3 and 1.4 since

$$\left(1 + \frac{1}{q}\right)^{-1} \cdot \left(1 - \frac{1}{q^2}\right) = \left(1 - \frac{1}{q}\right)$$

and

$$-\log\left(\prod_{q > q_k} \frac{q^2}{q^2 - 1}\right) \geq -\frac{2}{q_k}.$$

By Proposition 1.10, if the Riemann hypothesis is false, then there is a real number $0 < b < \frac{1}{2}$ such that there exist infinitely many natural numbers x for which $\log f(x) = \Omega_{+}(x^{-b})$. Actually Nicolas proved that $\log f(x) = \Omega_{\pm}(x^{-b})$, but we only need to use the notation Ω_{+} under the domain of the natural numbers. According to the Hardy and Littlewood definition, this would mean that

$$\exists k > 0, \forall y_0 \in \mathbb{N}, \exists y \in \mathbb{N}\ (y > y_0)\colon\ \log f(y) \geq k \cdot y^{-b}.$$

The previous inequality is also $\log f(y) \geq \left(k \cdot y^{-b} \cdot \sqrt{y}\right) \cdot \frac{1}{\sqrt{y}}$, but we notice that

$$\lim_{y \to \infty} \left(k \cdot y^{-b} \cdot \sqrt{y}\right) = \infty$$

for every possible values of $k > 0$ and $0 < b < \frac{1}{2}$. Now, this implies that

$$\forall y_0 \in \mathbb{N}, \exists y \in \mathbb{N}\ (y > y_0) \colon\ \log f(y) \geq \frac{1}{\sqrt{y}}.$$

Note that, the value of k is not necessary in the statement above. In this way, if the Riemann hypothesis is false, then there exist infinitely many natural numbers x such that $\log f(x) \geq \frac{1}{\sqrt{x}}$. In addition, if $\log f(x_0) \geq \frac{1}{\sqrt{x_0}}$ for some large enough natural number x_0, then $\log f(x_0) = \log f(q_k)$ and $\log f(q_k) \geq \frac{1}{\sqrt{x_0}}$ where q_k is the greatest prime number such that $q_k \leq x_0$ which could be also the largest prime factor of the superabundant number n at the same time. The reason is because of the equality of the following terms:

$$\prod_{q \leq x_0} \left(1 - \frac{1}{q}\right) = \prod_{q \leq q_k} \left(1 - \frac{1}{q}\right)$$

and

$$\theta(x_0) = \theta(q_k)$$

according to the definition of the Chebyshev function. Since $\frac{1}{\sqrt{x_0}} > \frac{1}{\sqrt{2 \cdot q_k}} > \frac{2}{q_k}$ for every large enough prime number q_k by Bertrand's postulate, then it would be infinitely many superabundant numbers n such that $\log g(n) > 0$ under the assumption that the Riemann hypothesis is false. □

3. Main Insight

This is the main insight.

Lemma 3.1. *The Riemann hypothesis is true whenever for each large enough superabundant numbers n, there exists another superabundant $n' > n$ such that*

$$R(n') \leq R(n).$$

Proof. Over the range of superabundant numbers n, we can state that

$$\lim_{n \to \infty} R(n) = \frac{e^{\gamma}}{\zeta(2)}$$

since

$$q_k \sim \log n \quad as \quad (n \to \infty)$$

and

$$\theta(q_k) \sim q_k \quad as \quad (q_k \to \infty)$$

where

$$\lim_{k \to \infty} R(N_k) = \frac{e^{\gamma}}{\zeta(2)}$$

and q_k is the largest prime factor of n: This is a consequence of putting together the Propositions 1.1, 1.6 and 1.8. By Lemma 2.1, if the Riemann hypothesis is false and the inequality

$$R(n') \leq R(n)$$

is satisfied for each large enough superabundant number n, then there exists an infinite subsequence of superabundant numbers n_i such that

$$R(n_{i+1}) \leq R(n_i),$$

$n_{i+1} > n_i$ and $\mathsf{Dedekind}(n_i)$ fails. This is a contradiction with the fact that

$$\liminf_{n \to \infty} R(n) = \lim_{n \to \infty} R(n) = \frac{e^{\gamma}}{\zeta(2)}.$$

By definition of the limit inferior for any positive real number ε, only a finite number of elements of $R(n)$ are less than $\frac{e^{\gamma}}{\zeta(2)} - \varepsilon$. This contradicts the existence of such previous infinite subsequence and thus, the Riemann hypothesis must be true. □

4. Main Theorem

This is the main theorem.

Theorem 4.1. *The Riemann hypothesis is true.*

Proof. By Lemma 3.1, the Riemann hypothesis is true whenever

$$R(n') \leq R(n)$$

is satisfied for large enough superabundant numbers $n' > n$. For every large enough superabundant number n with the largest prime factor q_{k-1}, we could take the greatest superabundant number $n' > n$ such that its largest prime factor is q_k. We are always able to find such superabundant number n' due to Proposition 1.5. That is the same as

$$\frac{\prod_{q \mid n'} \left(1 + \frac{1}{q}\right)}{\log \log n'} \leq \frac{\prod_{q \mid n} \left(1 + \frac{1}{q}\right)}{\log \log n}$$

and

$$\frac{\prod_{q \mid n'} \left(1 + \frac{1}{q}\right)}{\prod_{q \mid n} \left(1 + \frac{1}{q}\right)} \leq \frac{\log \log n'}{\log \log n}$$

which is

$$\frac{\log \log n'}{\log \log n} \geq \left(1 + \frac{1}{q_k}\right)$$

after of distributing the terms. By Proposition 1.7, we notice that

$$\left\lfloor \frac{\log q_k}{\log q_i} \right\rfloor \leq \nu_{q_i}(n')$$

for $2 \leq q_i \leq q_k$. That would be $\nu_{q_i}(n') > \frac{\log q_k}{\log q_i} - 1$ and so,

$$q_i^{\nu_{q_i}(n')+1} > q_k.$$

Consequently, we deduce that

$$n' \geq q_k \cdot n$$

since (q_{k-1}, q_k) is a pair of two consecutive primes such that $q_k > q_{k-1}$. Hence, it is enough to show that

$$\frac{\log\log(q_k \cdot n)}{\log\log n} \geq \left(1 + \frac{1}{q_k}\right).$$

By Proposition 1.2, we could obtain that

$$\begin{aligned}
\frac{\log\log(q_k \cdot n)}{\log\log n} &= \frac{\log\left(\log q_k + \log n\right)}{\log\log n} \\
&= \frac{\log\left((\log n) \cdot \left(1 + \frac{\log q_k}{\log n}\right)\right)}{\log\log n} \\
&= \frac{\log\log n + \log\left(1 + \frac{\log q_k}{\log n}\right)}{\log\log n} \\
&= 1 + \frac{\log\left(1 + \frac{\log q_k}{\log n}\right)}{\log\log n} \\
&\geq 1 + \frac{\frac{\frac{\log q_k}{\log n}}{\frac{\log q_k}{\log n}+1}}{\log\log n} \\
&= 1 + \frac{\frac{\log q_k}{\log q_k + \log n}}{\log\log n} \\
&= 1 + \frac{\log q_k}{(\log(q_k \cdot n)) \cdot \log\log n}.
\end{aligned}$$

We only need to prove that

$$1 + \frac{\log q_k}{(\log(q_k \cdot n)) \cdot \log\log n} \geq \left(1 + \frac{1}{q_k}\right).$$

That is equivalent to

$$q_k \cdot \log q_k \geq (\log(q_k \cdot n)) \cdot \log\log n.$$

However, the inequality

$$q_k \cdot \log q_k \geq (\log(q_k \cdot n)) \cdot \log\log n$$

holds for large enough superabundant numbers n and n' since we know that

$$n' \geq q_k \cdot n$$

and

$$q_k \cdot \log q_k \sim (\log n') \cdot \log\log n' \quad as \quad (n' \to \infty)$$

by Propositions 1.6 and 1.7. $\square$

5. Conclusions

Practical uses of the Riemann hypothesis include many propositions that are considered to be true under the assumption of the Riemann hypothesis and some of them that can be shown to be equivalent to the Riemann hypothesis [3]. Indeed, the Riemann hypothesis is closely related to various mathematical topics such as the distribution of primes, the growth of arithmetic functions, the Lindelöf hypothesis, the Large Prime Gap Conjecture, etc [3]. In general, a proof of the Riemann hypothesis could spur considerable advances in many mathematical areas [3].

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