Detecting Inclusions for O-convex hulls of Bichromatic Point Sets

Carlos Alegrı̃ a, David Orden, Carlos Seara and Jorge Urrutia

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Detecting inclusions for $\mathcal{O}$-convex hulls of bichromatic point sets

Carlos Alegría*[1], David Orden†[2], Carlos Seara‡[3], and Jorge Urrutia§[4]

[1]Universidad Nacional Autónoma de México
[2]Departamento de Física y Matemáticas, Universidad de Alcalá, Spain
[3]Departament de Matemàtiques, Universitat Politècnica de Catalunya, Spain
[4]Instituto de Matemáticas, Universidad Nacional Autónoma de México

Abstract

Let $\mathcal{O}$ be a set of lines through the origin. Given two disjoint sets of n red and n blue points in the plane, we study the problem of maintaining the subset of blue points contained in the $\mathcal{O}$-convex hull of the red point set while we rotate $\mathcal{O}$ around the origin. We describe efficient algorithms to solve the problem when $\mathcal{O}$ contains two lines. We consider the case where we simultaneously rotate both lines, and the case where one line is rotated while the second one is kept fixed.

1 Introduction

Restricted-orientation convexity [4, 5] is a non-traditional notion of convexity that studies geometric objects whose intersections with lines parallel to one from a given set $\mathcal{O}$ are connected. Since this notion of convexity was defined in the early eighties, several results of topological and combinatorial flavors can be found in the literature, as well as computational problems that are usually adaptations of well-known problems related to standard convexity [1, 4, 7, 9].

Despite all these results there are still fundamental questions to be answered. In this paper we explore the fundamental problem that, for illustrative purposes, we describe next in the context of standard convexity. Let R and B be two disjoint sets of n red and n blue points in the plane. We want to compute the subset of blue points contained in the convex hull of the red point set. We may then ask, for example, which is the set whose convex hull contains the largest (or smallest) subset of the other color, or ask for a particular condition, such as full containment. This problem can be trivially solved in $O(n \log n)$ time. See Figure 1.

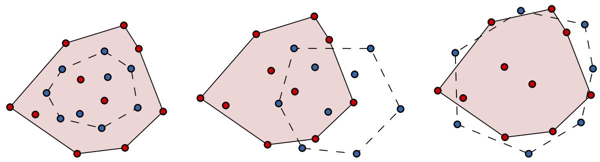

Figure 1: The set of blue points and the red convex hull. The cases of full containment (left), partial containment (center), and no containment (right).

Unlike the standard convex hull of a finite point set, the $\mathcal{O}$-convex hull (the restricted orientation version of the standard convex hull) is *orientation-dependent*: $\mathcal{O}$-convex hulls of the same point set at different orientations of the lines in $\mathcal{O}$ are non-congruent to each other. We thus translate the inclusion problem above to restricted orientations as the problem of computing the subset of blue points contained in the red $\mathcal{O}$-convex hull, and maintaining this set of points while we change the orientations of the lines in $\mathcal{O}$.

We restrict the problem to a set $\mathcal{O}$ of two lines passing through the origin. We first consider the case where the lines are orthogonal to each other and both are simultaneously rotated by an angle that goes from 0 to $\pi/2$. In this setting the $\mathcal{O}$-convex hull is known as the *rectilinear convex hull* [8]. We then consider the case where one line is kept fixed while the second one is rotated by an angle β that goes from 0 to π. In this setting the $\mathcal{O}$-convex hull is known as the *$\mathcal{O}_\beta$-convex hull* [1]. In both cases we solve the problem in optimal $O(n \log n)$ time and $O(n)$ space.

2 The rectilinear convex hull

In this section we assume that $\mathcal{O}$ is formed by two orthogonal lines. A *quadrant* is a translation of one of the four open regions that result from subtracting the lines of $\mathcal{O}$ from the plane. Given a point set P, a region in the plane is *P-free* if it contains no points of

*Email: calegria@uxmcc2.iimas.unam.mx
†Email: david.orden@uah.es. Research supported by Project MTM2017-83750-P of the Spanish Ministry of Science (AEI/FEDER, UE).
‡Email: carlos.seara@upc.edu. Research supported by the projects Gen. Cat. DGR 2017SGR1640 and MINECO MTM2015-63791-R.
§Email: urrutia@matem.unam.mx. Research supported in part by SEP-CONACYT 80268, PAPPIIT IN102117 Programa de Apoyo a la Investigación e Innovación Tecnológica UNAM.

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P. The rectilinear convex hull of P is defined as the set of points

$$\mathcal{RH}(P) = \mathbb{R}^2 \setminus \bigcup_{q \in \mathcal{Q}} q,$$

where $\mathcal{Q}$ denotes the set of all P-free quadrants of the plane. Let $\mathcal{RH}_\theta(P)$ denote the rectilinear convex hull of P computed after simultaneously rotating the lines in $\mathcal{O}$ around the origin by an angle of θ in the counter-clockwise direction. A P-free wedge with apex on a point $p \in P$ is *maximal*, if it is not contained in any other P-free wedge with apex on p. We refer to the aperture angle of a wedge as the *size* of the wedge.

Consider the disjoint point sets R and B. We adapt the definition of Θ-maximality from Avis et al. [2] to bichromatic point sets as follows.

Definition 1 *A blue point b is an unoriented $\pi/2$-maximal with respect to R if there is an R-free maximal wedge with apex on b and size at least $\pi/2$.*

By a straightforward adaptation of the results from Avis et al. [2] we obtain the following Lemma.

Lemma 2 *There is an algorithm to compute the set of all blue unoriented $\pi/2$-maximals with respect to R in $O(n \log n)$ time and $O(n)$ space.*

The algorithm mentioned in Lemma 2 receives as input the sets R and B, and reports all the blue unoriented $\pi/2$-maximals with respect to R. For each such maximal blue point b, the output also contains the (at most three) R-free maximal wedges with apex on b. Consider a blue point b that is an unoriented $\pi/2$-maximal with respect to R. Let w be one of the R-free maximal wedges with apex on b and size at least $\pi/2$, and w_0 be the wedge resulting from translating w so its apex lies on the origin. The *maximal arc* of b induced by w is the circular arc that results from the intersection between w_0 and $\mathbb{S}^1$ (the unit circle centered at the origin).

The origin splits the lines in $\mathcal{O}$ into four rays. We say that two rays are consecutive to each other if they are consecutive in their circular order around the origin. The following characterization is based on observations from Hurtado et al. [6].

Lemma 3 *For a given value of θ, a point $b \in B$ is contained in $\mathcal{RH}_\theta(R)$ if, and only if, no maximal arc of b with respect to R is intersected by two consecutive rays defined by the lines in $\mathcal{O}$. See Figure 2.*

2.1 The algorithm

Let B_θ denote the subset of blue points contained in $\mathcal{RH}_\theta(R)$. Our characterization naturally leads to an algorithm to maintain B_θ while θ is increased from 0 to $\pi/2$. We first compute all the R-free maximal wedges with apex on a blue point and size at least $\pi/2$. By Lemma 2 this can be done in $O(n \log n)$ time and $O(n)$ space. We then translate each wedge into a blue maximal arc with respect to R. There are at most three wedges per point of B and thus, each blue point has at most three maximal arcs. From the set of blue R-free maximal wedges we can therefore compute the set of $O(n)$ blue maximal arcs in $O(n)$ time. We store the set of all maximal arcs into a sorted circular list L in $O(n \log n)$ time.

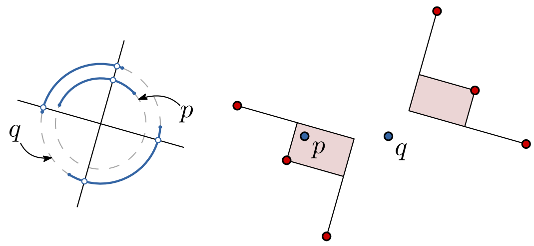

Figure 2: On the right, two blue points and $\mathcal{RH}_\theta(R)$. On the left, the lines of $\mathcal{O}$ and the blue maximal arcs. Instead of drawing $\mathbb{S}^1$, the maximal arcs of each blue point are drawn separately on concentric circles for the sake of clarity. Note that no consecutive rays intersect a maximal arc of p, and two pairs of consecutive rays intersect a maximal arc of q.

We now sweep $\mathbb{S}^1$ by counter-clockwise rotating $\mathcal{O}$ from 0 to $\pi/2$, while using L to predict the next value of θ where a ray passes over an endpoint of a maximal arc (an *intersection event*). We update B_θ at each intersection event according to the conditions from Lemma 3. These conditions are checked for a blue point b_i using an auxiliary variable n_i, that stores the number of consecutive pairs of rays intersecting a maximal arc of b_i. Consider an intersection event given by an endpoint of a maximal arc of a blue point b_i at an angle θ. The event is processed as follows: If a pair of consecutive rays start intersecting the maximal arc, then increase n_i by one. If a pair of consecutive rays stop intersecting the maximal arc, then decrease n_i by one. Leave n_i unchanged if neither of the above situations take place. Add b_i to B_θ if $n_i = 0$. Remove b_i from B_θ if $n_i = 1$.

Since we have $O(n)$ intersection events and we process each event in $O(1)$ time, the sweeping process takes $O(n)$ time. We obtain the following result.

Theorem 4 *The subset of blue points contained in $\mathcal{RH}_\theta(R)$ can be maintained while θ is increased from 0 to $\pi/2$ in $O(n \log n)$ time and $O(n)$ space.*

3 The $\mathcal{O}_\beta$-convex hull

For simplicity and without loss of generality, assume that one of the lines in $\mathcal{O}$ is the x-axis and the slope of the second one is equal to $\tan(\beta)$. We denote this set of lines with $\mathcal{O}_\beta$. An $\mathcal{O}_\beta$-*quadrant* is a translation of one of the four open regions that result from subtracting the lines of $\mathcal{O}_\beta$ from the plane. The $\mathcal{O}_\beta$-convex hull of P is the set of points

$$\mathcal{O}_\beta\mathcal{H}(P) = \mathbb{R}^2 \setminus \bigcup_{q \in \mathcal{Q}_\beta} q,$$

where $\mathcal{Q}_\beta$ is the set of all P-free $\mathcal{O}_\beta$-quadrants of the plane. Based on the results from Alegría et al. [1], the characterization from Section 2 can be easily translated to $\mathcal{O}_\beta$-convexity.

Definition 5 *For a given value of β, a blue point b is a β-maximal with respect to R if there is an R-free $\mathcal{O}_\beta$-quadrant with apex on b.*

Every blue point b is the apex of at most two R-free maximal wedges containing an horizontal ray. These wedges define at most four angles as shown in Figure 3. The point b is a β-maximal with respect to R if at least one of $\beta_1, \ldots, \beta_4$ is greater than β.

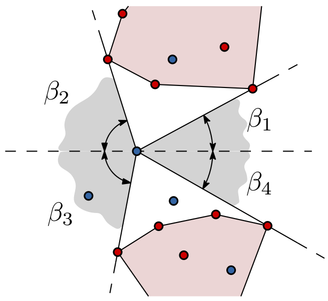

Figure 3: A blue point is the apex of two R-free maximal wedges that contain an horizontal ray.

We now rephrase Lemma 3 to β-convexity. The definition of the maximal arc of a blue point with respect to R is exactly the same, but considering only maximal wedges that contain an horizontal ray.

Lemma 6 *For a given value of β, a point $b \in B$ is contained in $\mathcal{O}_\beta\mathcal{H}(R)$ if, and only if, no maximal arc of b with respect to R is intersected by two consecutive rays. See Figure 4.*

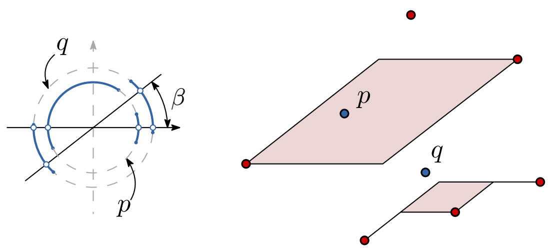

Figure 4: Inclusion of blue points in $\mathcal{O}_\beta\mathcal{H}(R)$.

3.1 The algorithm

Let B_β denote the subset of blue points contained in $\mathcal{O}_\beta\mathcal{H}(R)$. The algorithm to maintain B_β while β is increased from 0 to π is essentially the same we described in Section 2. By Lemma 6 an intersection event is processed exactly in the same way. The evident adaptation is the computation of the set of maximal arcs. To compute the set of R-free maximal wedges that contain an horizontal ray we use a sweep-line algorithm on the set $R \cup B$: Scan the set from top to bottom. When visiting a red point, use an on-line algorithm to construct the standard convex hull of the red visited points, one point at a time. When visiting a blue point b, compute the R-free wedges with apex on b bounded by an horizontal line through b and the tangents from b to the red convex hull. Repeat by scanning the points from bottom to top. See again Figure 3. This algorithm takes $O(n \log n)$ time and $O(n)$ space.

Theorem 7 *The subset of blue points contained in $\mathcal{O}_\beta\mathcal{H}(R)$ can be maintained while β is increased from 0 to π in $O(n \log n)$ time and $O(n)$ space.*

4 The lower bound

We show next that computing the subset of blue points contained in $\mathcal{RH}_\theta(R)$ for a fixed value of θ requires $\Omega(n \log n)$ time in the algebraic computation tree model. This result implies that the algorithm from Section 2 is time optimal. The proof can also be adapted to the $\mathcal{O}_\beta$-convex hull problem for the case where $\beta = \pi/2$.

Theorem 8 *Computing the subset of B contained in $\mathcal{RH}_\theta(R)$ for a fixed value of θ requires $\Omega(n \log n)$ time in the algebraic computation tree model.*

Proof. By reduction from the Integer Set Disjointness (ISD) problem, which has a lower bound of $\Omega(n \log n)$ time in the algebraic computation tree model [3]. Let $\mathcal{X} = \{x_1, \ldots, x_n\}$ and $\mathcal{Y} = \{y_1, \ldots, y_n\}$ be two sets of integers input to the ISD problem. Let $m \ll \min\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ and $M \gg \max\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$. In $O(n)$ time we transform $\mathcal{X}$ and $\mathcal{Y}$ into the input to our problem by producing the set of $2(n+1)$ red points

$$\begin{aligned} R = \{&r_i^{\text{NW}} = (x_i - \alpha, x_i + \alpha) \mid 1 \le i \le n\} \;\cup \\ &\{r_i^{\text{SE}} = (x_i + \alpha, x_i - \alpha) \mid 1 \le i \le n\} \;\cup \\ &\{r_m = (m+1, m+1), r_M = (M+1, M+1)\}, \end{aligned}$$

and the set of $2(n+1)$ blue points

$$\begin{aligned} B = \{&b_i^{\text{NE}} = (y_i + \beta, y_i + \beta) \mid 1 \le i \le n\} \;\cup \\ &\{b_i^{\text{SW}} = (y_i - \beta, y_i - \beta) \mid 1 \le i \le n\} \;\cup \\ &\{b_m = (m-1, m-1), b_M = (M-1, M-1)\} \end{aligned}$$

on the line with slope one, where $0 < \beta < \alpha < 1/2$. We use the values $\alpha = 1/3$ and $\beta = 1/6$ for our proof and, without loss of generality, we assume that $\theta = 0$. The construction is illustrated in Figure 5.

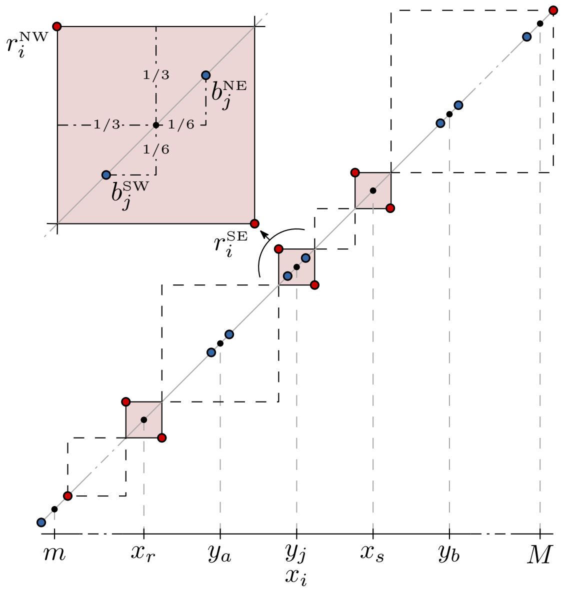

Figure 5: Transforming the sets of integers $\mathcal{X}$ and $\mathcal{Y}$ into the sets of points R and B. $\mathcal{RH}_0(R)$ is formed by $n+2$ disconnected components, two of which are single points. All the red points are vertices of $\mathcal{RH}_0(R)$.

If $\mathcal{X} \cap \mathcal{Y} \neq \emptyset$, then there are at least two integers $x_i \in \mathcal{X}$ and $y_j \in \mathcal{Y}$ such that $x_i = y_j$. In this case the points b_i^{NE} and b_i^{SW} are contained in $\mathcal{RH}_0(R)$. On the other hand, if $\mathcal{X}$ and $\mathcal{Y}$ are disjoint then every pair of integers $x_i \in \mathcal{X}$ and $y_j \in \mathcal{Y}$ is such that $x_i \neq y_j$. In this case no blue point is contained in $\mathcal{RH}_0(R)$. We thus have that $\mathcal{X} \cap \mathcal{Y} = \emptyset$ if, and only if, the subset of points of B contained in $\mathcal{RH}_0(R)$ is empty.

We have therefore reduced in linear time the ISD problem on $\mathcal{X}$ and $\mathcal{Y}$ to computing the subset of points of B contained in $\mathcal{RH}_0(R)$. $\square$

5 Concluding remarks

There are two aspects of our algorithms that are important to note. First of all, the sets R and B are not required to be balanced. If R and B contain n_r and n_b points respectively, the complexity of our algorithms is $O((n_r+n_b)\log(n_r+n_b))$ time and $O(n_r+n_b)$ space. Second, the particular case where no blue point is contained in the red $\mathcal{O}$-convex hull implies the existence of an $\mathcal{O}$-convex hull that separates the red points from the blue points. Therefore, our algorithms can be used to report, if any, all the angular intervals of separability in $O(n \log n)$ time and $O(n)$ space. It is not hard to show that even deciding the existence of a separability interval has a bound of $\Omega(n \log n)$ time, so the algorithms are time-optimal.

An interesting generalization of the rectilinear convex hull is to consider a set of k lines, $2 \leq k \leq n$, with arbitrary slopes. Let $r_1, \ldots, r_{2k}$ be the rays formed by the lines in $\mathcal{O}$ labeled in circular order. Let α_i be the size of the wedge bounded by r_{i+1} and r_{i+k}, where subindices are taken modulo $2k$, and $\Theta = \min\{\alpha_1, \ldots, \alpha_{2k}\}$. If $\Theta \geq \pi/2$, the $\mathcal{O}$-convex hull is defined in terms of wedges with size at least $\pi/2$. We can thus adapt the characterization from Section 2, and solve the problem when the lines in $\mathcal{O}$ are simultaneously rotated around the origin also in optimal $O(n \log n)$ time and $O(n)$ space. When $\Theta < \pi/2$, by the results from Avis et al. [2] the time complexity is increased by a factor of $1/\Theta$.

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