Riemann Hypothesis: Redheffer Matrix and Semi-Infinite Construction

Valerii Sopin

July 17, 2023

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Abstract. The Riemann Hypothesis is the conjecture that the Riemann zeta-function has its zeros only at the negative even integers and complex numbers with real part 1/2. Many consider it to be the most important unsolved problem in mathematics (the zeros of the Riemann zeta-function are the key to an analytic expression for the number of primes).

The Riemann Hypothesis is equivalent to the statement about the asymptotics of the Mertens function, the cumulative sum of the Möbius function. The Mertens function, in its turn, can be represented fairly simply as the determinant of a matrix (the Redheffer matrix) defined in terms of divisibility (square matrix, all of whose entries are 0 or 1), where the last can be considered as adjacency matrix, which is associated with a graph. Hence, for each graph it is possible to construct a statistical model.

The paper outlines the above and it presents an algebra (as is customary in the theory of conformal algebras), having manageable and painless relations (unitary representations of the $N = 2$ superVirasoro algebra appear). The introduced algebra is closely related to the fermion algebra associated with the statistical model coming from the infinite Redheffer matrice (the ith line can be viewed as a part of the thin basis of the statistical system on one-dimensional lattice, where any i consecutive lattice sites carrying at most $i-1$ zeroes). It encodes the bound on the growth of the Mertens function.

The Riemann zeta-function is a difficult beast to work with, that's why a way is to replace the divisibility.

1. Introduction

The most important feature of the Mertens function $M(n)$ is its connection with the Riemann Hypothesis [1]:

Proposition 1. *The Riemann Hypothesis is true if and only if it is true that* $M(n) = O(n^{1/2+\varepsilon})$ *for any* $\varepsilon > 0$.

Moreover, a weaker big-O statement about $M(n)$ leads to a weaker statement about the zeroes of the Riemann zeta-function:

Proposition 2. *If* $M(n) = O(n^{\alpha+\varepsilon})$ *for some fixed real* α *and any* $\varepsilon > 0$, *and* r *is a non-trivial zero of the Riemann zeta-function, then* $1 - \alpha \leq r \leq \alpha$.

The Redheffer matrix $A_n = \{a_{i,j}\}$ is defined by $a_{i,j} = 1$ if $j = 1$ or i divides j, and $a_{i,j} = 0$ otherwise. The determinants of the Redheffer matrices are tied to the Riemann Hypothesis through [2]:

Proposition 3. *The determinant of the* $n\times n$ *square Redheffer matrix is given by the Mertens function* $M(n)$.

A_n can be represented as a sum $A_n = C_n + D_n$: the matrix $D_n = \{d_{i,j}\}$ with $d_{i,j} = 1$ if and only if i divides j and the matrix $C_n = \{c_{i,j}\}$ with $c_{i,j} = 1$ if and only if $j = 1$ and $i \neq 1$. Hence, it is important to deal mainly with D_n (e.g. the Laplace expansion along the first column).

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A_6 = C_6 + D_6$$

Taking $n \to \infty$ and considering D_∞, the ith line of D_∞ is semi-infinite line, where after $i-1$ zeroes 1 is placed. Ideas from [3, 4] bring to mind that the ith line in D_∞ **can be viewed as a part of the thin basis of the statistical system on semi-infinite one-dimensional lattice, where** *any* i **consecutive lattice sites carrying at most** $i-1$ **zeroes**. It is a one-dimensional lattice, which defines a completely solvable model in Statistical Mechanics. Double grading in such model allows to trace the initial line.

$$\underbrace{\overbrace{0,\dots,0}^{r-1},1,\overbrace{0\dots,0}^{i-r}}_{i},\dots$$

The above is related to unitary representations of the $N = 2$ superVirasoro algebra [3]. The condition that i consecutive lattice sites carrying at most $i-1$ zeroes can be given by the functional form (as is customary in the theory of conformal algebras [3, 4]) imposed in addition to the standard Pauli principle for fermions.

In the paper it is suggested that the introduced later algebra, unifying the above ideas, encodes a bound on the growth of the Mertens function. Note that the determinant of an adjacency matrix counts the number of not-necessarily-connected-cycles (that is subgraphs being disjoint unions of connected cycles) passing through every vertex of the graph (it is what the determinant means in the context of a graph, see [5]; check also the Lindström-Gessel-Viennot lemma). The cycle is counted as -1 if the number of its components has different parity than the number of vertices of the graph, otherwise it is counted as 1. It is worth mentioning that if A is the adjacency matrix of a finite graph, then $\frac{1}{\det(I-At)}$ describes the "zeta-function" of the graph.

Remark 1. *The Lee-Yang theorem* [6] *is of considerable interest in the study of zeros. It says that the zeros of the partition function of a ferromagnetic Ising model are all on the unit circle (i.e. after changing the variable all zeros lie on a critical line with their real part equals to 0). So, a question now is: is the Riemann zeta-function the partition function of some spin system?*

On the contemporary state-of-the art of the Riemann Hypothesis, the interested reader is referred to [7, 8, 9, 10] and references therein.

The Riemann zeta-function is a difficult beast to work with, that's why a way is to replace the divisibility.

Remark 2. *One strategy to prove the Riemann Hypothesis is what is known as the Hilbert-Pólya conjecture. It involves finding a self-adjoint operator on a Hilbert space whose eigenvalues would be the ordinates of the zeros of the zeta function. Since the operator is self-adjoint these eigenvalues would be real. Note that in Quantum Mechanics a system is governed by a self-adjoint operator the Hamiltonian.*

Remark 3. *Robin's theorem states that*

$$\sigma(n) < e^{\gamma} n \log\log n$$

for all $n > 5040$ *if and only if the Riemann Hypothesis is true, where* γ *is the Euler–Mascheroni constant and* $\sigma(n)$ *is the divisor function given by* $\sigma(n) = \sum_{d|n} d$.

Moreover, the Riemann Hypothesis is true if and only if

$$\sum_{\rho} \frac{1}{|\rho|^2} = 2 + \gamma - \log(4\pi),$$

where γ *is the Euler–Mascheroni constant and* ρ *are the non-trivial zeros of the Riemann zeta-function* [11].

Remark 4. *The Riemann Hypothesis is the discrete version of Calabi-Yau theorem as solution of Ricci flat metric and the Riemann zeta-function can be interpreted by quantum gravity* [12].

Remark 5. *There exists a proof of the Riemann Hypothesis using absolute algebraic geometry over the field of one element* [13, 14].

Remark 6. *Hugh Montgomery and Freeman Dyson found that the pair-correlation function of the zeroes of the Riemann zeta-function resembled the pair-correlation function used to describe the energy levels of a heavy nucleus.*

2. The algebra

The fermion algebra [4] for the graph from the ith line of the adjacency matrix D_∞ is the following algebra of anti-commuting elements x^i_j, $j \in \mathbb{N}$:

$$\mathbb{C}[x^i_1, x^i_2, \dots]/(x^i_i x^i_{i\cdot k} = 0,\ k \in \mathbb{N}).$$

Let's obtain a two-dimensional model. It is the following algebra (which is not the fermion algebra [4] for the graph from the adjacency matrix D_∞, but it still encodes all information about divisibility) of anti-commuting elements x^i_j, $i, j \in \mathbb{N}$ (swapping two rows changes the sign of the determinant):

$$\sum_{i=1}^{\infty} (\mathbb{C}[x^i_1, x^i_2, \dots]/(x^i_i x^i_{i\cdot k} = 0,\ k \in \mathbb{N})).$$

Let's replace the conditions $x^i_i x^i_{i\cdot k} = 0$ by $x^i_j x^i_{j+1} \dots x^i_{j+i-1} = 0$ (i.e. <u>*any*</u> i consecutive lattice sites carrying at most $i-1$ zeroes; the role is changed: $0 \leftrightarrow 1$).

It is important to highlight that double grading exists: the number of elements and the sum of their indexes in a monomial. This allows to trace initial lines of D_∞.

Let's denote generating functions $X^i(z) = \sum_{j=1}^{\infty} x^i_j z^{-j}$ and let's determine the deformation (dimensions are preserved [3, 4]) of the algebra generated by anti-commuting elements, satisfying the relations below

$$\partial^{i-1} X^i(z) \partial^{i-2} X^i(z) \dots \partial X^i(z) X^i(z) = 0.$$

The described is related to unitary representations of the $N = 2$ superVirasoro algebra, see [3]. Note the relations among string theory, four-dimensional $N = 4$ supersymmetric Yang-Mills theory and the Riemann Hypothesis, see [9].

Remark 7.
$$\sum_{k=1}^{n} e^{2\pi\sqrt{-1}d\frac{k}{n}} = \begin{cases} n & \text{if } n \text{ divides } d \\ 0 & \text{otherwise} \end{cases}$$

Remark 8. *The Hadamard's inequality is*
$$|\det(M)| \le \prod_{j=1}^{n} \|m_j\|_2,$$
where m_j denotes the jth column of $M = \{m_{i,j}\}$, which is a $n \times n$ matrix.

Moreover,
$$|\det(M)| \le \prod_{i,j} (1 + |m_{ij}|).$$

3. Concluding remarks

We don't have a good clear approach to the Riemann Hypothesis, but it has so many unclear approaches! Nevertheless, the presented approach will lead to interesting mathematics as it is about two-dimensional Statistical Mechanics and Superconformal Algebras.

Finally I want to mention developments in the statistical theory of L-functions based on Random Matrix Theory [15]. These have their beginnings in Montgomery's pair correlation conjecture [16].

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Email address: vvs@myself.com