Fuzzy goal programming approach for solving fuzzy multi-objective stochastic linear programming problem

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***Abstract*—This paper deals with the multi-objective chance constrained programming, where the right hand side of the constraints are normally distributed and the objective functions with fuzzy numbers coefficients. A fuzzy goal programming approach is developed for the corresponding deterministic problem by defining membership values and aspiration levels. The advantage of the proposed approach is the decision-maker's role only in estimating the efficient solution to avoid or at least limit the influences of his/ her knowledge incomplete about the studied problem. A numerical example is given in the utility of the paper to illustrate the applied and the efficiency of the approach.**

***Index Terms*—Chance constrained programming, Normal distribution, Joint probability distribution, fuzzy numbers, Fuzzy programming, Goal programming, Optimal compromise solution**

I. Introduction

ONE of the difficulties occur in the mathematical programming (MP) applications is that the parameters in the problem formulation are not crisp but fluctuating and uncertain [1]. In most of the real life problems in MP, the parameters are considered as random variables. The branch of MP which deals with the theory and methods for the solution of conditional extremum problems under in complete information about the random parameters is called "stochastic programming" [2]. When the random variables are normally distributed with known means and variances, Conini [3] investigated an algorithm for solving stochastic goal programming. Based on the concept of chance constraints introduced by Charnes & Cooper [4], and using probabilistic goals, Sullivan and Fitzsimmoms [5] suggested an algorithms. Most of the problems in applied mathematics may belong any one of the following classes [6]:

- Descriptive Problems, in which information is processed about the investigated event, some laws of the events being introduced by others, that is occurred with the help of mathematical methods.
- Optimization Problems in which from a set of feasible solutions, an optimal solution is chosen.

Besides the previous division of applied mathematics problems, they may be classified into ordinary and stochastic problems. Several mathematical methods have been applied in the process of the solution of the stochastic problem. It has been applied in the management science [2]. Leclereq [7] investigated an interactive approach for stochastic programming.

Two major approaches to stochastic programming [8], [9] are organized as:

- Chance constrained programming;
- Two-stage programming.

The chance constrained programming (CCP) technique is one which can be used to solve problems involving chance constraints i.e., constraints having finite probability of being violated. The CCP has been used in several directions and various applications too. CCP models can be converted into the corresponding deterministic mathematical problem. In order to solve this kind of problems, a stochastic simulation based genetic algorithm in which the stochastic simulation was used to check the feasibility of solution has been proposed by Iwamura and Liu [10]. Shen et al. [11] used chance constraints for the risk of undesirable random outcomes. Du *et al.* [12] presented a fuzzy multi-objective programming model that minimizes the risk, travel time and fuel consumption for the transportation problem. Under uncertain delivered time, Kalinina *et al.* [13] investigated a multi Objective chance constrained programming for matching of goods and transports. Yang *et al.* [14] applied CCP for optimizing spare parts inventory. Masoud et al. [15], [16] applied CP as an effective approach to solve rail problems as a deterministic mathematical application. In many scientific areas, such as system analysis and operations research, a model has to be set up using data which is only approximately known. Fuzzy sets theory, introduced by Zadeh [17], makes this possible. Dubois and Prade [18] extended the use of algebraic operations on real numbers to fuzzy numbers by the use a fuzzification principle. Tanaka and Asal [19] formulated a fuzzy linear programming (FLP) problem to obtain a reasonable solution under consideration of the ambiguity of parameters. Rommelfanger *et al.* [20] introduced an interactive method for solving multi-objective linear optimization problem, where coefficients of the objectives and/ or of the constraints are known exactly but

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imprecisely. Zhao *et al.* [21] studied the complete solution set for fuzzy linear programming problems included fuzzy and non-fuzzy equality and inequality constraints. Hamadameen [22] proposed a technique for solving fuzzy MOLP problem in which the objective functions coefficients are triangular fuzzy numbers. Kiruthiga and Loganathan [23] reduced the Fuzzy MOLP problem to the corresponding ordinary one using the ranking function and hence solved it using the fuzzy programming technique. Zimmermann [24] developed fuzzy programming approach for solving multi-objective linear programming problem. Masoud et al. [25] and Liu and Kozan [26] solved an advanced sugarcane railway scheduling optimisation model by job shop scheduling and metaheuristic algorithms. Masoud et al. [27], [28] and Liu et al. [29] developed a multi-objective mixed integer programming (MIP) formulation to design a customized scheduling optimizer for the sugarcane rail transport system. Also, Yan et al. [30] and Li et al. [31] solved solved many industrial applications based on three graph-based constructive algorithms for multi-stage scheduling and treating multi-regional factors in the industrial green development.

In this paper, the proposed approach is developed to solve enormous types of optimization problems with of different types of parameters in accessible way comparing with algebraic methods, where selecting the weights reduce the effort to obtain satisfactory solution. The research will be extended in different topics of operations research in future work.

The rest of the paper is organized as follows: In section 2, some preliminaries need in the paper are introduced. In section 3, multi-objective a stochastic programming problem is formulated and defined. In section 4, a fuzzy goal programming approach for solving the corresponding deterministic problem is given. The stability set of the first kind without differentiability is determined in section 4. In section 5, a solution algorithm for solving the problem is proposed. In section 6, a numerical example is developed for illustration. Finally some concluding remarks are reported in section 7.

II. Preliminaries

In order to discuss our problem conveniently, we introduce fuzzy numbers (Kauffmann and Gupta [32]; Fottemps and Roubens [33])

Definition 1: (Kauffmann and Gupta [32]). A fuzzy number $\widetilde{a}$ is a mapping:
$\mu_{\widetilde{a}} : R \to [0,1]$, with the following properties:
(i) $\mu_{\widetilde{a}}$ is an upper semi-continuous membership function;
(ii) $\widetilde{a}$ is a convex fuzzy set, i.e., $\mu_{\widetilde{a}}(\lambda x^1 + (1-\lambda)x^2) \geq \min\{\mu_{\widetilde{a}}(x^1), \mu_{\widetilde{a}}(x^2)\}$ for all $x^1, x^2 \in R, 0 \leq \lambda \leq 1$;
(iii) $\widetilde{a}$ is normal, i.e., $\exists x_0 \in R$ for which $\mu_{\widetilde{a}}(x_0) = 1$;
(iv) Supp$(\widetilde{a}) = \{x \in R : \mu_{\widetilde{a}}(x) > 0\}$ is the support of $\widetilde{a}$, and its closure cl (supp$(\widetilde{a})$) is a compact set.

It is assumed that $F_0(R)$ is the set of all fuzzy numbers.

A function, usually denoted by "L" or "R", is a reference function of a fuzzy number if and only if

1. $L(x) = L(-x)$,
2. $L(0) = 1$,
3. L is non-increasing on $[0, \infty]$

A convenient representation of fuzzy numbers in the $L-R$ flat fuzzy number which is defined as

$$\mu_{\widetilde{A}}(x) = \begin{cases} L((A^- - x)\eta) & x \leq A^-, \eta > 0 \\ R((x - A^+)\beta) & x \geq A^+, \beta > 0 \\ 1 & \text{elsewhere} \end{cases} \quad (1)$$

where, $A^- \leq A^+$; $[A^-, A^+]$ is the core of $\widetilde{A}$; $\mu_{\widetilde{A}}(x) = 1, \forall x \in [A^-, A^+]$; A^-, A^+ are the lower and upper modal values of $\widetilde{A}$, respectively; and $\eta > 0, \beta > 0$ are the left-hand and right-hand spreads (Roubens [34]).

Remark 1: A flat fuzzy number is denoted by $\widetilde{A} = (A^-, A^+, \eta, \beta)_{LR}$
Among the various type of fuzzy numbers, trapezoidal fuzzy numbers, denoted by $\widetilde{A} = (A^-, A^+, \eta, \beta)_{LR}$, are of the greatest importance (Roubens [34]).

The main concept of comparison of fuzzy numbers is based on the compensation of areas determined by the membership functions(Baldwin and Guild [35], and Nakamura [36]).

Let $\widetilde{p}, \widetilde{q}$ be fuzzy numbers and $S_L(\widetilde{p}, \widetilde{q}), S_R(\widetilde{p}, \widetilde{q})$ be the areas determined by their membership functions according to

$$S_L(\widetilde{p}, \widetilde{q}) = \int_{I(\widetilde{p}, \widetilde{q})} (\inf \widetilde{p}_a - \inf \widetilde{q}_a) d\alpha, \quad (2)$$

$$S_R(\widetilde{p}, \widetilde{q}) = \int_{S(\widetilde{p}, \widetilde{q})} (\sup \widetilde{p}_a - \sup \widetilde{q}_a) d\alpha, \quad (3)$$

where $I(\widetilde{p}, \widetilde{q}) = \{\alpha : \inf \widetilde{p}_a \geq \inf \widetilde{q}_a\} \subset [\theta, 1], \theta > 0$, and $S(\widetilde{p}, \widetilde{q}) = \{\alpha : \sup \widetilde{p}_a \geq \sup \widetilde{q}_a\} \subset [\theta, 1], \theta > 0$.

The degree to which is defined (Roubens [34]) as

$$C(\widetilde{p}, \widetilde{q}) = S_L(\widetilde{p}, \widetilde{q}) - S_L(\widetilde{q}, \widetilde{p}) + S_R(\widetilde{p}, \widetilde{q}) - S_R(\widetilde{q}, \widetilde{p}) \quad (4)$$

Proposition 1: (Roubens [34]). Let $\widetilde{p}$ and $\widetilde{q}$ be $L-R$ fuzzy numbers with parameters $(p^-, p^+, \eta, \beta), (q^-, q^+, \gamma, \delta)$ and reference functions $(L_{\widetilde{p}}, R_{\widetilde{p}}), (L_{\widetilde{q}}, R_{\widetilde{q}})$, where all reference functions are invertible. Then $\widetilde{p}(\geq)\widetilde{q}$ if and only if $\sup \widetilde{p}_{\alpha_{\widetilde{p}}, R} + \inf \widetilde{p}_{\alpha_{\widetilde{p}}, L} \geq \sup \widetilde{q}_{\alpha_{\widetilde{q}}, R} + \inf \widetilde{q}_{\alpha_{\widetilde{q}}, L}$.

If $k = \widetilde{p} \otimes \widetilde{q}$ then $\alpha_{k.R} = R_k(\int_0^1 R_k^{-1}(\alpha) d\alpha), \alpha_{k.L} = L_k(\int_0^1 L_k^{-1}(\alpha) d\alpha)$.

Remark 2: $\widetilde{p} \geq \widetilde{q}$ if and only if $p^- + p^+ + \frac{1}{2}(\beta - \eta) \geq q^- + q^+ + \frac{1}{2}(\delta - \gamma)$.

Definition 2: The associated real number p corresponding to $\widetilde{p} = (p^-, p^+, \eta, \beta)$ is defined as $R(\widetilde{p}) = \hat{p} = p^- + p^+ + \frac{1}{2}(\beta - \eta)$.

Let $F(R)$ be the set of all trapezoidal fuzzy numbers.

III. Problem definition and solution concepts

Consider a multi-objective chance constrained programming (MOCCP) problem with joint probability distributions constraints and $L-R$ fuzzy numbers in the objective functions coefficients as

$$\min \widetilde{Z}^{(k)}(x) = \sum_{j=1}^{n} \widetilde{c}_j^{(k)} \otimes x_j, k = 1, 2, ..., K \quad (5)$$

subject to:

$$\Pr(\sum_{j=1}^{n} a_{ij} x_j \geq b_i) \geq 1 - \theta_i, i = 1, 2, ..., m, \quad (6)$$

$$x_j \geq 0, j = 1, 2, ..., n. \quad (7)$$

where $\widetilde{c}_j = (c_j^-, c_j^+, \omega_j, \xi_j) \in F(R), j = 1, 2, ..., n; b_i(i = 1, 2, ..., m)$ are normal random variables with means and variances are known and $\theta_i \in (0, 1)$ is specified probability. It is assumed that:

- the normal random variables are independent, and
- the decision variables are crisp.

Definition 3: The x^* which satisfies the conditions (6) and (7), is called a fuzzy pseudo-random efficient solution of the problem (5)-(7) if and only if there is no x^0 such that $\widetilde{Z}^{(k)}(x^0)(\leq)\widetilde{Z}^{(k)}(x^*)$, for all k; and $\widetilde{Z}^{(k)}(x^0)(\neq)\widetilde{Z}^{(k)}(x^*)$, for some k.

Let $E(b_i)$, and $\sigma(b_i)$ be the means and the standard deviations of the normal random variables b_i. Hence, inequality (6) can be rewritten as

$$\prod_{i=1}^{n} \Pr(\sum_{j=1}^{n} a_{ij}x_j \geq b_i) \geq 1 - \theta_i \tag{8}$$

Also, (5) can be simplified in the following form

$$\prod_{i=1}^{n} \Pr(\frac{b_i - E(b_i)}{\sigma(b_i)}(\leq)\frac{\sum_{j=1}^{n} a_{ij}x_j - E(b_i)}{\sigma(b_i)}) \geq 1 - \theta_i \tag{9}$$

where $\frac{b_i - E(b_i)}{\sigma(b_i)}$ follows standard normal distribution with mean and variance are zero, and one; respectively.

So,

$$\prod_{i=1}^{n} \phi(\frac{\sum_{j=1}^{n} a_{ij}x_j - E(b_i)}{\sigma(b_i)}) \geq 1 - \theta_i \tag{10}$$

where ϕ represents the cumulative distribution function corresponding to the standard normal variable. Let

$$\tau = \frac{\sum_{j=1}^{n} a_{ij}x_j - E(b_i)}{\sigma(b_i)}, \tag{11}$$

and

$$\phi(\tau) = w_i, i = 1, 2, ..., m, \tag{12}$$

Then,

$$\prod_{i=1}^{n} w_i \geq 1 - \theta_i \tag{13}$$

For the standard normal distribution, the probability density function is defined as

$$\phi(\tau_i) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\tau_i} dz \tag{14}$$

By putting $h = z^2/2$ into (14), we obtain

$$\phi(\tau_i) = \frac{1}{2}(\frac{\gamma(\frac{1}{2}, \frac{\pi_i^2}{2})}{\Gamma(\frac{1}{2})} - 1) \tag{15}$$

where

$$\gamma(\frac{1}{2}, \frac{\pi_i^2}{2}) = \int_0^x h^{-1/2}e^{-h}dh \tag{16}$$

$$= e^{-\frac{\tau_i^2}{2}}(\frac{\tau_i^2}{2})^{1/2} \sum_{s=0}^{\infty} \frac{\Gamma(1/2)}{\Gamma(3/2 + s)}(\frac{\tau_i^2}{2})^s \tag{17}$$

From (17) to (12), and simplified we get

$$\sum_{s=0}^{\infty} \frac{\tau_i^{(2s-1)}}{\prod_{n=0}^{s}(2n+1)} = \sqrt{\frac{\pi}{2}}(2w_i + 1)e^{\frac{\tau_i^2}{2}} \tag{18}$$

It follows that the series may be expanded as
$\sum_{s=0}^{\infty} \frac{\tau_i^{(2s-1)}}{\prod_{n=0}^{s}(2n+1)} = \tau_i(1 + \frac{1}{3}\tau_i^2 + \frac{1}{25}\tau_i^4 + ... + ...)$
$\leq \tau_i(1 + \frac{1}{3}\tau_i^2 + (\frac{1}{3})^2\tau_i^4 + ... + ...)$
$= \tau_i(\frac{1}{1-\frac{1}{3}\tau_i^2}), \quad \tau_i^2 \neq 3 \quad (i.e., \tau_i^3 < 3)$

From the above expansion into (18), we obtain

$$\tau_i(\frac{1}{1 - \frac{1}{3}\tau_i^2})e^{-\frac{\tau_i^2}{2}} \geq \sqrt{\frac{\pi}{2}}(2w_i + 1) \tag{19}$$

Based on the previous and by using the definition 2, the equivalent ordinary problem corresponding to the problem (5)-(7) can be reformulated as

$$\begin{aligned} &\min Z^{(k)}(x) = R(\textstyle\sum_{j=1}^{n} \widetilde{c}_j^{(k)} \otimes x_j) \\ &= (\textstyle\sum_{j=1}^{n} c_j^{(k)} x_j); k = 1, 2, ..., K \end{aligned} \tag{20}$$

Subject to

$$\begin{cases} \tau_i(\frac{1}{1-\frac{1}{3}\tau_i^2})e^{-\frac{\tau_i^2}{2}} \geq \sqrt{\frac{\pi}{2}}(2w_i + 1); \\ \prod_{i=1}^{n} w_i \geq 1 - \theta_i; \\ \sum_{j=1}^{n} a_{ij}x_j = \tau_i\sigma(b_i) + E(b_i); \\ 0 \leq w_i \leq 1, i = 1, 2, ..., m; \\ x_j \geq 0, j = 1, 2, ..., n; \end{cases} \tag{21}$$

IV. Fuzzy goal programming approach for solving MOCCP problem

Bellman and Zadeh [37] introduced three basic concepts: fuzzy goal (,fuzzy constraints (, and fuzzy decision and explored the application of these concepts to decision- making processes under fuzziness

The fuzzy decision is the fuzzy set defined as:

$$D = G \cap C \tag{22}$$

The fuzzy decision is characterized by its membership function:

$$\mu_D(x) = \min(\mu_G(x), \mu_C(x)) \tag{23}$$

Consider the following multi-objective linear programming problem

$$\text{(MOLP)}\ Z^{(k)}(x) = (Z^{(1)}(x), Z^{(2)}(x), ..., Z^{(k)}(x))^T \tag{24}$$

Subject to

$$x \in X = \{x \in R^n : Ax \leq b, x \geq 0\} \tag{25}$$

Having elicited membership functions $\mu_k(Z^{(k)}(x)), k = 1, 2, ..., K$, from the decision maker (DM) for each of the objective functions$Z^{(k)}(x)$, the above MOLP problem can be converted into the fuzzy optimization problem

$$\max_{x\in X}(\mu_1(Z^{(1)}(x)),\mu_2(Z^{(2)}(x)),...,\mu_k(Z^{(k)}(x)))^T \quad (26)$$

The fuzzy decision or the minimum operator of Bellman and Zadeh [37])

$$\max \max_{k=1,2,...,K}(\mu_1(Z^{(1)}(x)),\mu_2(Z^{(2)}(x)),...,\mu_k(Z^{(k)}(x))) \quad (27)$$

By introducing an auxiliary variable v, the fuzzy multi-objective decision making problem can be transformed into the following well known linear or nonlinear programming model as

$$\begin{aligned} & \max v && (28)\\ \text{Subject to} \quad & v \le \mu_k(Z^{(k)}(x)), k=1,2,...,K && (29)\\ & x\in X && (30)\\ & 0\le v\le 1 && (31)\end{aligned}$$

A. Linear membership function

The linear membership function for MOLP problem is given by:

$$\mu_k(Z^{(k)}(x)) = \begin{cases} 0 & Z^{(k)}(x)\le L_k\\ \frac{Z^{(k)}(x)-L_k}{U_k-L_k} & L_k\le Z^{(k)}(x)\le U_k\\ 1 & Z^{(k)}(x)\ge U_k\end{cases} \quad (32)$$

Where U_k, and L_k are the upper bound and the lower bound of the objective functions k, respectively which are calculated as follows:

$$U_k=(Z^{(k)})^{\max}=\max_{x\in X} Z^{(k)}(x),$$
$$\text{and} L_k=(Z^{(k)})^{\min}=\min_{x\in X} Z^{(k)}(x), k=1,2,...,K$$

Using the linear membership function IV-A, problem (28)-(31) becomes

$$\begin{aligned} & \max v && (33)\\ \text{Subject to} \quad & v\le \frac{Z^{(k)}(x)-L_k}{U_k-L_k}, k=1,2,...,K && (34)\\ & x\in X && (35)\\ & 0\le v\le 1 && (36)\end{aligned}$$

To formulate problem (38)-(36) as a goal programming (Sakawa [38]), let the following positive and negative deviations are to be considered as:

$$Z^{(k)}(x)+d_k^l-d_k^u=H^k, k=1,2,...,K \quad (37)$$

Where, $H^k$is the aspiration level of the objective functions k Problem (38)-(36) with these goals (37) can be reformulated as follows

$$\begin{aligned} & \max v && (38)\\ \text{Subject to} \quad & v\le \frac{Z^{(k)}(x)-L_k}{U_k-L_k}, k=1,2,...,K && (39)\\ & Z^{(k)}(x)+d_k^l-d_k^u=H^k, k=1,2,...,K && (40)\\ & x_j, d_k^l, d_k^u\ge 0, j=1,2,...,n; 0\le v\le 1 && (41)\end{aligned}$$

Thus, problem (20)-(21) becomes

$$\max v \quad (42)$$

Subject to

$$\begin{cases} v(U_k-L_k)\le (Z^{(k)}(x)-L_k)m, k=1,2,...,K;\\ Z^{(k)}(x)+d_k^l-d_k^u=H^k, k=1,2,...,K;\\ \tau_i(\frac{1}{1-\frac{1}{3}\tau_i^2})e^{-\frac{\tau_i^2}{2}}\ge\sqrt{\frac{\pi}{2}}(2w_i+1);\\ \prod_{i=1}^n w_i\ge 1-\theta_i;\\ \sum_{j=1}^n a_{ij}x_j=\tau_i\sigma(b_i)+E(b_i);\\ 0\le w_i\le 1, i=1,2,...,m;\\ x_j\ge 0, j=1,2,...,n; \text{and} \tau_i \text{are unrestricted in sign}\end{cases} \quad (43)$$

V. Solution procedure

In this section, a solution procedure for solving problem (5)-(7) can be summarized as in the following steps:

Step 1: Converting the given problem (5)-(7) into the corresponding ordinary problem (20)-(21),

Step 2: Evaluate the objective function at the solution and determine the lower bound, the upper bound,

Step 3: Construct fuzzy goal programming by defining linear membership function (IV-A) and also initial aspiration level defined as in (37) to develop the problem (45)-(46),

Step 4: Solve problem (45)-(46) to get optimal compromise solution.

Step 5: Stop.

From our discussion above, the features of the solution procedure are: 1. It applies a more convenient and other types of membership functions may be used. 2. It could be extended to treat a fuzzy goal programming with objective functions coefficients in vague, imprecise, etc. 3. It can directly applied to solve different topics of operations Research

VI. Numerical Example

Consider the following problem

$$\min \widetilde{Z}^{(1)}(x)=\widetilde{c_1}^{(1)}x_1+\widetilde{c_2}^{(1)}x_2+\widetilde{c_3}^{(1)}x_3 \quad (44)$$
$$\min \widetilde{Z}^{(2)}(x)=\widetilde{c_1}^{(2)}x_1+\widetilde{c_2}^{(2)}x_2+\widetilde{c_3}^{(2)}x_3 \quad (45)$$

subject to

$$Pr(2x_1+x_2+2x_3\ge b_1; x_1+2x_2+4x_3\ge b_2)\ge 0.90 \quad (46)$$
$$x_1,x_2,x_3\ge 0 \quad (47)$$

Where,

$$\widetilde{c_1}^{(1)}=(0,1,1,3), \widetilde{c_2}^{(1)}=(1,1,1,3), \widetilde{c_3}^{(1)}=(2,4,2,4),$$

$$\widetilde{c_1}^{(2)} = (0,1,1,1), \widetilde{c_2}^{(2)} = (0,1,1,3), \widetilde{c_3}^{(2)} = (1,2,2,4),$$
$$E(b_1) = 7, E(b_2) = 6, \sigma(b_1) = 3, \sigma(b_2) = 2.$$

Step1: The deterministic problem corresponding to the problem (44)-(47) is shown below as

$$\min \widetilde{Z}^{(1)}(x) = 2x_1 + 3x_2 + 6x_3$$
$$\min \widetilde{Z}^{(2)}(x) = x_1 + 2x_2 + 4x_3$$

Subject to

$$2x_1 + x_2 + 2x_3 - 3\tau_1 = 6,$$
$$x_1 + 2x_2 + 4x_3 - 4\tau_1 = 7,$$
$$\sqrt{\frac{18}{\pi}}\tau_1 e^{-\frac{\tau_1^2}{2}} \geq (3 - \tau_1^2)(2w_1 + 1),$$
$$\sqrt{\frac{18}{\pi}}\tau_2 e^{-\frac{\tau_2^2}{2}} \geq (3 - \tau_2^2)(2w_2 + 1),$$
$$w_1 w_2 = 0.90,$$
$$0 \leq w_1, w_2 \leq 1,$$
$$x_1, x_2, x_3, w_1, w_2 \geq 0,$$
$$\tau_1, \text{and} \tau_2 \text{are unrestricted in sign.}$$

Step2: The solution of the proposed problem in Step 1 is shown in Table 1, where two models with single objective functions are solved.

TABLE I
SOLUTION OF EACH INDIVIDUAL OBJECTIVE FUNCTION

$Model1$	Objective function(Z^1)	21.5994
	x_1	2.7473
	x_2	4.2524
	x_3	0.5579
	τ_1	1.6210
	τ_2	10.3493
	w_1	0.9000
	w_2	1.000
$Model2$	Objective function(Z^2)	18.4838
	x_1	2.7473
	x_2	1.9925
	x_3	1.6879
	τ_1	1.6210
	τ_2	13.2784
	w_1	0.9000
	w_2	1.000

Step3: Using the membership function and the goals, the fuzzy goal programming for the problem can be formulated as:

$$\max v$$

Subject to

$$\begin{cases} 2x_1 + 3x_2 + 6x_3 - 0.0001v \geq 21.5994, \\ x_1 + 2x_2 + 4x_3 - 0.0001 \geq 13.4838, \\ x_1 + 2x_2 + 4x_3 + d_1^l - d_1^u = 21.5995, \\ x_1 + 2x_2 + 4x_3 + d_2^l + d_2^u = 13.4837, \\ \tau_1(\frac{1}{1-\frac{1}{3}\tau_i^2})e^{-\frac{\tau_1^2}{2}} \geq \sqrt{\frac{\pi}{2}}(2w_1 + 1); \\ \tau_2(\frac{1}{1-\frac{1}{3}\tau_i^2})e^{-\frac{\tau_2^2}{2}} \geq \sqrt{\frac{\pi}{2}}(2w_2 + 1); \\ w_1 w_2 \geq 0.90, \\ 2x_1 + x_2 + 2x_3 - 3\tau_1 = 6, \\ x_1 + 2x_2 + 4x_3 - 4\tau_1 = 7, \\ 0 \leq w_i, v \leq 1, i = 1, 2, \\ x_j \geq 0, j = 1, 2, 3, \\ \text{and} \tau_1, \tau_2 \text{are unrestricted in sign.} \end{cases} \tag{48}$$

Step4. Solve the above problem through the problem (45)-(46), we have

TABLE II
SOLUTION OF THE PROBLEM (29)

	Objective function $\widetilde{Z_1}$	(5.3683,11.8047,8.1155,20.6573)
	Objective function $\widetilde{Z_2}$	(1.8446,8.1155,8.1155,15.1629)
v=0.9398	x_1	2.7472
	x_2	1.6791
	x_3	1.8446
	τ_1	1.6208
	τ_2	1.7321
	w_1	0.3205
	w_2	0.4282
	d_1^l	8.8146
	d_1^u	0.6997
	d_2^l	0
	d_2^u	0

VII. CONCLUSIONS

In this paper, the fuzzy multi-objective stochastic linear programming problem has been studied. These probabilistic parameters have been normally distributed having joint probability distribution with known means and variances. A fuzzy goal programming approach is applied to the equivalent deterministic problem for obtaining optimal compromise solution. The developed approach is formulated as non-linear programming models and coded by Matlab Language programming. Computational experiments are demonstrated to provide insightful decisions.

This work opens the field to many future avenues of research with a particular emphasis on transportation and health optimization under uncertainties. There are several benefits to formulate the transport system optimization problem as a fuzzy multi-objective stochastic linear programming, namely to improve the performance considering the arrival and departure times as stochastic elements; reducing the total costs associated with the proposed trips; and, limit negative effects that this system can have on the travelers. Furthermore, in the health systems optimization, the proposed approach will increase the patient flow rates through the hospital to optimize the treatment times and patients waiting times.

References

[1] C. Veeramani, C. Duraisamy, and A. Nagoorgani, "Solving fuzzy multi-objective linear programming problems with linear membership functions," *Australian Journal of Basic and Applied Sciences*, vol. 5, no. 8, pp. 1163–1171, 2011.

[2] I. Stanch-Minasian, "A research bibliography in stochastic programming," *Oper. Res.*, vol. 24, pp. 1078–1119, 1976.

[3] B. Contini, "A stochastic approach to goal programming," *Operations Research*, vol. 16, no. 3, pp. 576–586, 1968.

[4] A. Charnes and W. W. Cooper, "Chance-constrained programming," *Management science*, vol. 6, no. 1, pp. 73–79, 1959.

[5] R. Sullivan and J. Fitzsimmoms, "A goal programming model for readiness and optimal deployment of resources," *Socio-Economic Planning Science*, vol. 12, pp. 215–220, 1978.

[6] V. Kolbin, *Stochastic Programming*. Reidel Publishing Company, Dordrecht, 1977.

[7] J.-P. Leclercq, "Stochastic programming: an interactive multicriteria approach," *European Journal of Operational Research*, vol. 10, no. 1, pp. 33–41, 1982.

[8] A. Goicoechea, D. R. Hansen, and L. Duckstein, "Multiobjective decision analysis with engineering and business applications," John Wiley & Sons, Tech. Rep., 1982.

[9] N. Kambo, *Mathematical Programming Techniques*. Affiliated East-West Press Pvt. Ltd., 1984.

[10] K. Iwamura and B. Liu, "A genetic algorithm for chance constrained programming," *Journal of Information and Optimization sciences*, vol. 17, no. 2, pp. 409–422, 1996.

[11] S. Shen, M. Kurt, and J. Wang, "Chance-constrained programming models and approximations for general stochastic bottleneck spanning tree problems," *INFORMS Journal on Computing*, vol. 27, no. 2, pp. 301–316, 2015.

[12] J. Du, L. Yu, and X. Li, "Fuzzy multi-objective chance-constrained programming model for hazardous materials transportation," *International Journal of General Systems*, vol. 45, no. 3, pp. 286–310, 2016.

[13] M. Kalinina, L. Olsson, and A. Larsson, "A multi objective chance constrained programming model for intermodal logistics with uncertain time," *International Journal of Computer Science Issues*, vol. 10, no. 6, pp. 35–44, 2013.

[14] Y. Yang, K. Wei, R. Kang, and S. Wang, "Multi-objective chance constrained programming of spare parts based on uncertainty theory," *IEEE Access*, vol. 6, pp. 50 049–50 054, 2018.

[15] M. Masoud, E. Kozan, and G. Kent, "A constraint programming approach to optimise sugarcane rail operations," *In Taha, Z (Ed.) Proceedings of the 11th Asia Pacific Industrial Engineering and Management Systems Conference*, vol. 6, no. 2, pp. 1–7, 2010.

[16] M. Masoud, G. Kent, E. Kozan, and S. Liu, "A new constraint programming approach for optimising a coal rail system," *Journal of Optimization Letters*, vol. 11, no. 4, pp. 725–738, 2017.

[17] L. A. Zadeh, "Fuzzy sets," *Information Control*, vol. 8, pp. 338–353, 1965.

[18] D. Dubois and H. Prade, *Possibility Theory: An Approach to Computerized Processing of Uncertainty*. Plenum, New York., 1980.

[19] H. Tanaka, "A formulation of fuzzy linear programming problem based on comparison of fuzzy numbers," *Control and cybernetics*, vol. 13, pp. 185–194, 1984.

[20] H. Rommelfanger, R. Hanuscheck, and J. Wolf, "Linear programming with fuzzy objectives," *Fuzzy sets and systems*, vol. 29, no. 1, pp. 31–48, 1989.

[21] R. Zhao, R. Govind, and G. Fan, "The complete decision set of the generalized symmetrical fuzzy linear programming problem," *Fuzzy Sets and Systems*, vol. 51, no. 1, pp. 53–65, 1992.

[22] O. Hamadameen, "A noval technique for solving multi-objective linear programming problems," *Aro-The Scientific Journal of Koya University*, vol. 5, no. 2, pp. 1–8, 2017.

[23] M. Kiruthiga and C. Loganathan, "Fuzzy multi-objective linear programming problem using membership function," *International journal of science, engineering, and technology, applied sciences*, vol. 5, no. 8, pp. 1171–1178, 2015.

[24] H.-J. Zimmermann, "Fuzzy programming and linear programming with several objective functions," *Fuzzy sets and systems*, vol. 1, no. 1, pp. 45–55, 1978.

[25] M. Masoud, E. Kozan, and G. Kent, "Hybrid metaheuristic techniques for optimising sugarcane rail operations," *International Journal of Production Research*, vol. 53, no. 9, pp. 2569–2589, 2015.

[26] S. Liu and E. Kozan, "A hybrid shifting bottleneck procedure algorithm for the parallel-machine job-shop scheduling problem," *Journal of the Operational Research Society*, vol. 63, pp. 168–182, 2012.

[27] M. Masoud, G. Kent, E. Kozan, and S. Liu, "A new multi-objective model to optimise rail transport scheduler," *Journal of Transportation Technologies*, vol. 6, no. 2, pp. 86–98, 2016.

[28] M. Masoud, E. Kozan, G. Kent, and S. Liu, "An integrated approach to optimise sugarcane rail operations," *Computers Industrial Engineering*, vol. 98, pp. 211–220, 2016.

[29] S. Liu, E. Kozan, M. Masoud, Y. Zhang, and F. Chan, "Job shop scheduling with a combination of four buffering constraints," *International Journal of Production Research*, vol. 56, no. 9, pp. 3274–3293, 2018.

[30] P. Yan, S. Liu, C. Yang, and M. Masoud, "A comparative study on three graph-based constructive algorithms for multi-stage scheduling with blocking," *Journal of Industrial Management Optimization*, vol. 15, no. 1, pp. 221–233, 2019.

[31] W. Li, R. Wang, J.and Chen, S. Xi, Y.and Liu, F. Wu, M. Masoud, and X. Wu, "Innovation-driven industrial green development: The moderating role of regional factors," *Journal of Cleaner Production*, vol. 222, pp. 344–354, 2019.

[32] A. Kaufmann and M. M. Gupta, *Fuzzy Mathematical Models in Engineering and Management Science*. Elsevier Science Publishing Company INC., New York., 1988.

[33] P. Fortemps and M. Roubens, "Ranking and defuzzification methods based on area compensation," *Fuzzy sets and systems*, vol. 82, no. 3, pp. 319–330, 1996.

[34] M. Roubens, "Inequality constraints between fuzzy numbers and their use in mathematical programming," in *Stochastic versus fuzzy approaches to multiobjective mathematical programming under uncertainty*. Springer, 1990, pp. 321–330.

[35] J. Baldwin and N. Guild, "Comparison of fuzzy sets on the same decision space," *Fuzzy sets and Systems*, vol. 2, no. 3, pp. 213–231, 1979.

[36] K. Nakamura, "Preference relations on a set of cost in source and fuzzy utilities," *Fuzzy Sets and Systems*, vol. 20, pp. 147–162, 1986.

[37] R. E. Bellman and L. A. Zadeh, "Decision-making in a fuzzy environment," *Management science*, vol. 17, no. 4, pp. B–141, 1970.

[38] M. Sakawa, *Fuzzy sets and interactive multiobjective optimization*. New York, USA: Plenium Press, 1993.