Distributed Neurodynamic Approach for Optimal Allocation with Separable Resource Losses

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Abstract. To solve the optimal allocation problem with separable resource losses, this paper proposes a neurodynamic approach based on multi-agent system. By using KKT condition, the nonlinear coupling equality constraint in the original problem is equivalently transformed into a convex coupling inequality constraint. Then, with the help of finite-time tracking technology and fixed-time projection method, a neurodynamic approach is designed and its convergence is strictly proved. Finally, simulation results verify the effectiveness of the proposed neurodynamic approach.

Keywords: Distributed neurodynamic approach · Optimal allocation problem · Separable resource losses.

1 Introduction

Solving optimal allocation problems plays a pivotal role in achieving optimal resource utilization and load balancing. Rational allocation of resources holds the potential to enhance the efficiency of extensive systems, including power distribution networks [8], integrated energy system [16], and sensor network [20], thereby reducing overall costs while ensuring equitable resource allocation among nodes.

In general, the network equipped with n nodes is responsible for allocating a certain amount of resources and achieving the objective of minimizing the overall production cost of the entire system. According to the actual situation, the production cost incurred by each node is determined by its own device configuration and the local amount of resource allocation. Considering the resource constraints, the optimal allocation problem can be mathematically formulated as follows:

$$\begin{aligned} \min \ & \sum_{i=1}^{n} f_i(x_i) \\ \text{s.t.} \ & \sum_{i=1}^{n} x_i = D, \ x_i \in \mathcal{X}_i \end{aligned} \tag{1}$$

where $x_i \in \mathbb{R}$ denotes the actual allocation of node i, $f_i : \mathbb{R} \to \mathbb{R}$ is the production cost function of node i, $D \in \mathbb{R}$ is the given amount of resources, and $\mathcal{X}_i$ is the local constraints imposed on node i. Recently, as an optimized parallel computing model, the brain-like nonlinear dynamic system in the recurrent neural network, the so-called neurodynamic approach, has been known. With the help of the parallel computing and information interaction ability of multi-agent systems, which are not available in traditional optimization numerical methods, designing distributed neurodynamic approaches based on multi-agent systems has become a prevalent method for optimal allocation problems (1), such as [2, 3, 6, 10, 19].

In the optimal allocation problem (1), the equality constraint is related to the resources of all nodes in the network, so it is actually a global constraint, which is essentially different from local constraints. The coupling of global constraints usually makes it difficult to design distributed neurodynamic approaches. For example, the neurodynamic approaches which can effectively solve the distributed optimization problem with only local constraints [1, 12] cannot be directly used to solve the optimal allocation problem (1). Additionally, global inequality constraints often exist in practical applications, which makes it more challenging to maintain the distributed manner of neurodynamic approaches. Some related studies have been given in [11, 13].

Although the aforementioned results advance the technology of neurodynamic approaches, there are few neurodynamic approaches that can deal with resource losses. Resource loss usually destroys the convexity of coupling equality constraint, which brings great difficulties to searching for the solution of resource allocation problems. To handle this challenge, a neurodynamic approach is proposed to solve the optimal allocation problem with separable resource losses by combining projection operator and symbolic function in this paper. Compared with the existing literature, the main contributions of this paper are as follows:

- In contrast to [2, 3, 6, 10, 11, 13, 8], the optimal allocation problems in this paper take the resource losses into account. Thus, the discussed optimal allocation problem is more general and challenging.
- The states of agents along the proposed neurodynamic approach finally reach the exact optimal solution to the optimal allocation problem rather than the optimal solution set (see [9, 18]), which ensures a better convergence property.

2 Preliminaries and Problem Description

2.1 Preliminaries

Graph Theory. In the paper, the distributed neurodynamic approach is based on the multi-agent system, and the system operation is closely related to the communication network. The n agents in the system can be regarded as n nodes on the network, and the communication network can be represented by the topology graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{A})$, which is composed of node set $\mathcal{V} = \{1, 2, \cdots, n\}$, edge set $\mathcal{E} = \{(i, j) \subseteq \mathcal{V} \times \mathcal{V} : a_{ij} > 0\}$, and adjacency matrix $\mathcal{A} = [a_{ij}]_{n\times n}$. For the multi-agent system, $(i, j) \in \mathcal{E}$ means that agent i and agent j can exchange information. If the adjacency matrix $\mathcal{A}$ is symmetric, the graph $\mathcal{G}$ is said to be undirected. An undirected graph $\mathcal{G}$ is said to be connected if there are links between every two different nodes. More details can be found in [7].

Convex Analysis. A subset $\Omega \subseteq \mathbb{R}^n$ is called a convex set if for any $x \in \Omega$, $y \in \Omega$, and $\lambda \in [0,1]$, it has $(1-\lambda)x + \lambda y \in \Omega$. For the convex set Ω, if $f((1-\lambda)x + \lambda y) \leqslant (1-\lambda)f(x) + \lambda f(y)$ holds for all $x,\ y \in \Omega$ and $\lambda \in [0,1]$, then the function $f : \Omega \to \mathbb{R}$ is called a convex function. Furthermore, if there is $w > 0$ such that $f((1-\lambda)x + \lambda y) \leqslant (1-\lambda)f(x) + \lambda f(y) - \frac{w}{2}\lambda(1-\lambda)\|x-y\|^2$ holds for all $x,\ y \in \Omega$ and $\lambda \in [0,1]$, then the function $f : \Omega \to \mathbb{R}$ is called a w-strongly convex function. For a point x outside the convex set Ω, the distance from the projected point $\mathcal{P}_\Omega(x)$ to x is the minimum distance from x to Ω. Its mathematical definition is as follows

$$\mathcal{P}_\Omega(x) := \arg\min_{y \in \Omega} \|x - y\|.$$

The projection of x on a convex set Ω is unique and satisfies $\langle x - \mathcal{P}_\Omega(x), y - \mathcal{P}_\Omega(x)\rangle \leqslant 0$ for all $y \in \Omega$.

2.2 Problem Description

In the field of power grid, there are usually various forms of losses. For example, according to the operating conditions, the copper loss or core loss of the generator can reach one-tenth of the power generation [4], which can not be ignored. In addition, the transmission and distribution of power will also cause losses, which will greatly reduce the overall efficiency. Following the symbol of the optimal allocation problem (1), the optimal allocation problem with separable resource losses is

$$\begin{aligned} \min\ & \sum_{i=1}^{n} f_i(x_i) \\ \text{s.t.}\ & \sum_{i=1}^{n} x_i = D + \Psi(x),\ x_i \in \mathcal{X}_i \end{aligned} \tag{2}$$

where $\Psi(x) = \sum_{i=1}^{n} \Psi_i(x_i)$ is the separable resource losses, $\mathcal{X}_i = [x_i^L, x_i^U]$ is the local box constraint for node i, which is defined with the lower bound x_i^L and the upper bound x_i^U.

Assumption 1 *The functions in the optimal allocation problem* (2) *are defined as* $f_i(x_i) = \alpha_i x_i^2 + \beta_i x_i + \gamma_i$ *and* $\Psi_i(x_i) = a_i x_i^2 + b_i x_i$ *with* $\alpha_i > 0$*,* $\beta_i > 0$*, and* $a_i > 0$ *for all* $i \in \mathcal{V} := \{1, 2, \cdots, n\}$*.*

Assumption 2 *For all* $i \in \mathcal{V}$*, it holds* $1 - b_i - 2a_i x_i^U > 0$ *and the total amount* D *of resource satisfies* $\sum_{i=1}^{n}(x_i^L - \Psi_i(x_i^L)) \leqslant D \leqslant \sum_{i=1}^{n}(x_i^U - \Psi_i(x_i^U))$*.*

Assumption 3 *The communication graph $\mathcal{G}$ of the multi-agent system is undirected and connected.*

Remark 1. It should be noted that Assumption 1 implies the objective function and loss function of the optimal allocation problem (2) are both strongly convex. But the optimal allocation problem (2) is not necessarily a convex optimization problem, because the equality constraint is not affine. Assumption 2 guarantees that the feasible region of the optimal allocation problem (2) is nonempty and $1 - b_i - 2a_i x_i^U > 0$ follows $1 - \frac{\mathrm{d}\Psi(x)}{\mathrm{d}x_i} > 0$, which refers to the resource losses cannot exceed total resource supply.

3 Main Results

3.1 Approach Description

In view of the nonlinearity of equality constraints, it is impossible to directly use the related theory of convex optimization to solve the optimal solution of the optimal allocation problem (2). Denote $D = \sum_{i=1}^{n} d_i$, the following conclusion shows that solving the optimal allocation problem (2) can be transformed into solving a convex optimization problem with global inequality constraints.

Theorem 1. *Under Assumptions 1 and 2, the optimal solution of the following problem*

$$\begin{aligned} &\min\ F(x) = \sum_{i=1}^{n} f_i(x_i) \\ &s.t.\ \sum_{i=1}^{n} (d_i + \Psi_i(x_i) - x_i) \leqslant 0,\ x_i \in \mathcal{X}_i \end{aligned} \tag{3}$$

is the optimal solution of the optimal allocation problem (2).

Proof. Let $x^* = (x_1^*, x_2^*, \cdots, x_n^*)^{\mathrm{T}}$ be the optimal solution of the problem (3), then there are optimal Lagrange multipliers $\theta > 0$, $u^* = (u_1^*, u_2^*, \cdots, u_n^*)^{\mathrm{T}}$, and $v^* = (v_1^*, v_2^*, \cdots, v_n^*)^{\mathrm{T}}$ satisfying

$$\begin{aligned} &\nabla f_i(x_i^*) - \theta(1 - \nabla\Psi_i(x_i^*)) + u_i^* - v_i^* = 0 \\ &\theta \sum_{i=1}^{n} (d_i + \Psi_i(x_i^*) - x_i^*) = 0,\ \theta > 0 \\ &\sum_{i=1}^{n} (d_i + \Psi_i(x_i^*) - x_i^*) \leqslant 0 \\ &u_i^*(x_i^* - x_i^U) = 0,\ u_i^* \geqslant 0,\ x_i^* - x_i^U \leqslant 0 \\ &v_i^*(x_i^L - x_i^*) = 0,\ v_i^* \geqslant 0,\ x_i^L - x_i^* \leqslant 0 \end{aligned} \tag{4}$$

for all $i \in \mathcal{V}$.

Next, we prove that $\sum_{i=1}^{n}(d_i + \Psi_i(x_i^*) - x_i^*) = 0$. Suppose that this equation does not hold, then it implies $\theta = 0$ and $\sum_{i=1}^{n}(d_i + \Psi_i(x_i^*) - x_i^*) < 0$ from (4). According to Assumption 1, one has $\nabla f_i(x_i^*) = 2\alpha_i x_i^* + \beta_i > 0$ for all $i \in \mathcal{V}$. Hence, for any $i \in \mathcal{V}$,

$$v_i^* = \nabla f_i(x_i^*) - \theta(1 - \nabla\Psi_i(x_i^*)) + u_i^* = \nabla f_i(x_i^*) + u_i^* > 0.$$

This reveals that $x_i^* = x_i^L$ and $\sum_{i=1}^{n}(d_i+\Psi_i(x_i^L)-x_i^L) = \sum_{i=1}^{n}(d_i+\Psi_i(x_i^*)-x_i^*) < 0$, which contradicts $\sum_{i=1}^{n} x_i^L - \Psi(x_i^L) \leqslant D \leqslant \sum_{i=1}^{n} x_i^U - \Psi(x_i^U)$ in Assumption 2. It can be obtained that $D = \sum_{i=1}^{n} x_i^* - \Psi(x^*)$, that is, x^* is a feasible solution of the problem (2). Combining with the definition of x^*, we get that x^* is a optimal solution of problem (2).

Based on the Lagrangian multiplier method, here comes a useful lemma.

Lemma 1. *[17] Under Assumptions 1 and 2, $x^* = (x_1^*, x_2^*, \cdots, x_n^*)^{\mathrm{T}}$ is an optimal solution to the problem* (3) *if and only if there exists $x^* \in \mathbb{R}^n$ and $y^* \in \mathbb{R}$ such that*

$$\begin{aligned} x^* =& \mathcal{P}_{\mathcal{X}}(x^* - \nabla F(x^*) - \nabla g(x^*)y^*) \\ y^* =& [y^* + g(x^*)]^+ \end{aligned} \tag{5}$$

where $\mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_n$, $\nabla F(x^*) = (\nabla f_1(x_1^*), \nabla f_2(x_2^*), \cdots, \nabla f_n(x_n^*))^{\mathrm{T}}$, $\nabla g(x^*) = (\nabla g_1(x_1^*), \nabla g_2(x_2^*), \cdots, \nabla g_n(x_n^*))^{\mathrm{T}}$, and $g(x^*) = \sum_{i=1}^{n} g_i(x_i^*)$.

To tackle the convex optimization problem (3), we introduce a multi-agent system composed with n agents to represent the n nodes in the network, and for each agent $i \in \mathcal{V}$, design the following neurodynamic approach

$$\begin{cases} \dot{x}_i(t) = -x_i(t) + \mathcal{P}_{\mathcal{X}_i}\Big(x_i(t) - \nabla f_i(x_i(t)) - \nabla g_i(x_i(t))[y_i(t) + z_i(t)]^+\Big) \\ \dot{y}_i(t) = J_i(t) + |J_i(t)|\mathrm{sign}(K_i(t)) - \mathrm{sig}(K_i(t))^{\mu} - \mathrm{sig}(K_i(t))^{\nu} \\ \dot{z}_i(t) = k_1 \displaystyle\sum_{j\in\mathcal{N}_i} \mathrm{sign}\big(z_j(t) - z_i(t)\big) + n\nabla g_i(x_i(t))^{\mathrm{T}}\dot{x}_i(t) \\ J_i(t) = k_2 \displaystyle\sum_{j\in\mathcal{N}_i} \mathrm{sign}\big(y_j(t) - y_i(t)\big) - \frac{1}{2}y_i(t) + \frac{1}{2}[y_i(t) + z_i(t)]^+ \\ K_i(t) = [y_i(t)]^+ - y_i(t) \end{cases} \tag{6}$$

with initial values $x_i(0) \in \mathcal{X}_i$ and $z_i(0) = ng_i(x_i(0))$, where $g_i(x_i) = d_i + \Psi_i(x_i) - x_i$, $[s]^+ = \max\{s, 0\}$, $\mathrm{sig}(s)^{\mu} = \mathrm{sign}(s)|s|^{\mu}$, and $\mathcal{N}_i$ is the neighbor set of agent i. In the neurodynamic approach (6), control parameters $k_1 > 0$, $k_2 > 0$, $0 < \mu < 1$, and $\nu > 1$ are usually used to adjust its convergence rate.

Remark 2. It is worth noting that the existence of the solution of the neurodynamic approach (6) has been discussed in [5] and [13]. Furthermore, although the total number n of agents is needed in the neurodynamic approach (6), it can be easily determined distributively by employing the finite-time tracking technique introduced in [14].

3.2 Convergence Analysis

In this section, the finite-time tracking technique and the fixed-time projection method constructed in the neurodynamic approach (6) are used to make distributed estimates of the required global information, and the finite-time consistency of the auxiliary variables and the convergence of the proposed neurodynamic approach are discussed.

Lemma 2. *Under Assumptions 1-3, starting from $x_i(0) \in \mathcal{X}_i$, there is $T_1 > 0$ such that $y_i(t) \geqslant 0$ and $x_i(t)$ is bounded.*

Proof. **Step 1.** We prove that there is T_1 such that $y_i(t) \geqslant 0$ when $t \geqslant T_1$. Consider the following Lyapunov function $V_1(t) = (y_i - [y_i]^+)^2$. Clearly, the derivative of V_1 along the neurodynamic approach (6) is

$$\begin{aligned} \dot{V}_1 &= 2(y_i - [y_i]^+)^{\mathrm{T}} \dot{y}_i \\ &= 2(y_i - [y_i]^+)^{\mathrm{T}} J_i(t) - 2|J_i(t)||y_i - [y_i]^+| - 2|y_i - [y_i]^+|^{1+\mu} - 2|y_i - [y_i]^+|^{1+\nu} \\ &\leqslant -2V_1^{\frac{1+\mu}{2}} - 2V_1^{\frac{1+\nu}{2}}. \end{aligned} \tag{7}$$

Therefore, from Lemma 1 in [15], there is $T_1 \leqslant \frac{2}{2(1-\mu)} + \frac{1}{2(\nu-1)}$ such that $y_i(t) \geqslant 0$ when $t \geqslant T_1$.

Step 2. Let us show that $x_i(t) \in \mathcal{X}_i$ for all $t \geqslant 0$. Denote $p_i(t) = \mathcal{P}_{\mathcal{X}_i}(x_i(t) - \nabla f_i(x_i(t)) - \nabla g_i(x_i(t))[y_i(t) + z_i(t)]^+)$, then $p_i(t) \in \mathcal{X}_i$. From (6), we have

$$x_i(t) = \mathrm{e}^{-t} x_i(0) + (1 - \mathrm{e}^{-t}) \int_0^t p_i(s) \frac{\mathrm{e}^s}{\mathrm{e}^t - 1} \mathrm{d}s.$$

Since $\int_0^t \frac{\mathrm{e}^s}{\mathrm{e}^t - 1} \mathrm{d}s = 1$, $x_i(0) \in \mathcal{X}_i$ and $\mathcal{X}_i$ is convex set, it holds that $x_i(t) \in \mathcal{X}_i$, for any $t \geqslant 0$ and $i \in \mathcal{V}$. Hence, $x_i(t)$ is bounded from the boundedness of $\mathcal{X}_i$.

According to the conclusion of Lemma 2, we assume that M satisfies $\|x_i(t)\| \leqslant M$, and then the neurodynamic approach (6) is reduced as

$$\begin{cases} \dot{x}_i(t) = -x_i(t) + \mathcal{P}_{\mathcal{X}_i}\Big(x_i(t) - \nabla f_i(x_i(t)) - \nabla g_i(x_i(t))[y_i(t) + z_i(t)]^+\Big) \\ \dot{y}_i(t) = k_2 \sum\limits_{j \in \mathcal{N}_i} \mathrm{sign}\big(y_j(t) - y_i(t)\big) - \dfrac{1}{2} y_i(t) + \dfrac{1}{2}[y_i(t) + z_i(t)]^+ \\ \dot{z}_i(t) = k_1 \sum\limits_{j \in \mathcal{N}_i} \mathrm{sign}\big(z_j(t) - z_i(t)\big) + n \nabla g_i(x_i(t))^{\mathrm{T}} \dot{x}_i(t) \end{cases} \tag{8}$$

when $t \geqslant T_1$. Since $\nabla g_i(x_i)$ $(i \in \mathcal{V})$ are bounded based on Assumption 1, there is $M_1 > M$ such that $n^2 \|\nabla g_i(x_i(t))^{\mathrm{T}} \dot{x}_i(t)\| \leqslant M_1$. Thus, from the Lemma 3.2 in [13], if $k_1 > 2M_1$, then we have $T_2 > T_1$ satisfying $z_i(t) = \frac{1}{n} \sum_{j=1}^n z_j(t)$ $\forall i \in \mathcal{V}$ when $t \geqslant T_2$. From $z_i(0) = n g_i(x_i(0))$, it follows that

$$z_i(t) = \sum_{i=1}^n g_i(x_i(t))$$

for all $i \in \mathcal{V}$ when $t \geqslant T_2$. Thus, it is induced that $z_i(t)$ is bounded due to the definition of g_i in Assumption 1. Similarly, we can find $M_2 > 0$ such that $\| -\frac{1}{2}y_i(t) + \frac{1}{2}[y_i(t) + z_i(t)]^+\| \leqslant \frac{1}{2}\|z_i\| \leqslant M_2$ when $t \geqslant T_2$, which implies that if $k_2 > 2M_2$, there exists $T_3 > T_2$ holding

$$y_i(t) = \sum_{j=1}^{n} y_j(t)$$

for all $i \in \mathcal{V}$ when $t \geqslant T_3$.

From Lemma 2 and the above discussion, one can get that the values of the control parameters μ and ν in the neurodynamic approach (6) determine the upper bound of the time for $y_i(t)$ greater than zero, and the difference of k_1 and k_2 will also change the maximum time for $z_i(t)$ and $y_i(t)$ to reach consensus. Therefore, the value of the control parameters will affect the convergence rate of the neurodynamic approach (6) to some extent. In practical application, the value of the control parameter is usually selected and adjusted according to the parameter adjustment experience.

Theorem 2. *Under Assumptions 1-3, starting from initial values* $x_i(0) \in \mathcal{X}_i$ *and* $z_i(0) = ng_i(x_i(0))$*, if* $k_l > 2M_l$ $(l = 1, 2)$*, then the trajectories* $x_i(t)$ *of neurodynamic approach* (6) *asymptotically converges to the optimal solution to the problem* (2)*.*

Proof. Let $x = (x_1, x_2, \cdots, x_n)^{\mathrm{T}}$, $y = \sum_{i=1}^n y_i$, and $z = \sum_{i=1}^n z_i$, then when $t \geqslant T_3$, $z(t) = g(x(t))$ and the neurodynamic approach (8) can be rewritten as

$$\begin{cases} \dot{x}(t) = -x(t) + \mathcal{P}_{\mathcal{X}}\Big(x(t) - \nabla F(x(t)) - \nabla g(x(t))[y(t) + g(x(t))]^+\Big) \\ \dot{y}(t) = -\dfrac{1}{2}y(t) + \dfrac{1}{2}[y(t) + g(x(t))]^+ \end{cases} \tag{9}$$

Define $\Theta = \{(x^*, y^*) \in \mathbb{R}^{n+1} : (x^*, y^*) \text{ satisfies } (5)\}$, then according to Lemma 1, $(x^*, y^*) \in \Theta$ is the equilibrium point of neurodynamic approach (9), and x^* is the optimal solution to the problem (3).

Letting $h(x, y) = F(x) + \frac{1}{2}\|[y + g(x)]^+\|^2$, it is obvious that $h(x, y)$ is convex with respect to $(x, y) \in \mathbb{R}^n \times \mathbb{R}$. For $t \geqslant T_3$, consider a Lyapunov function

$$V(x,y) = h(x,y) - h(x^*,y^*) - (x-x^*, y-y^*)^{\mathrm{T}}\nabla h(x^*,y^*) + \frac{1}{2}\|x-x^*\|^2 + \frac{1}{2}\|y-y^*\|^2$$

with $(x^*, y^*) \in \Theta$, then we have $y^* = [y^* + g(x^*)]^+$ and $V(x, y) \geqslant \frac{1}{2}\|x - x^*\|^2 + \frac{1}{2}\|y - y^*\|^2$. According to $x^* = \mathcal{P}_{\mathcal{X}}(x^* - \nabla F(x^*) - \nabla g(x^*)y^*)$, denote $\hat{y} = [y + g(x)]^+$ and $\hat{y}^* = [y^* + g(x^*)]^+$, it gets

$$\begin{aligned} \dot{V} = &-\langle \nabla F(x) + \nabla g(x)\hat{y} - \nabla F(x^*) - \nabla g(x^*)\hat{y}^* + x - x^*, \\ &\quad x - x^* + x^* - \mathcal{P}_{\mathcal{X}}(x - \nabla F(x) - \nabla g(x)\hat{y})\rangle + \frac{1}{2}\langle \hat{y} - y, \hat{y} + y - 2y^*\rangle \\ = &-\langle x - x^*, \nabla F(x) + \nabla g(x)\hat{y} - \nabla F(x^*) - \nabla g(x^*)\hat{y}^*\rangle - \|x - x^*\|^2 \\ &- \langle \nabla F(x) + \nabla g(x)\hat{y} - \nabla F(x^*) - \nabla g(x^*)\hat{y}^* + x - x^* \\ &\quad x^* - \mathcal{P}_{\mathcal{X}}(x - \nabla F(x) - \nabla g(x)\hat{y})\rangle - \frac{1}{2}\|\hat{y} - y\|^2 + \langle \hat{y} - y, \hat{y} - y^*\rangle \end{aligned} \tag{10}$$

Since

$$\begin{aligned}\|x-x^*\|^2 =& \|x-\mathcal{P}_{\mathcal{X}}(x-\nabla F(x)-\nabla g(x)\hat{y})+\mathcal{P}_{\mathcal{X}}(x-\nabla F(x)-\nabla g(x)\hat{y})-x^*\|^2\\ =& \|x-\mathcal{P}_{\mathcal{X}}(x-\nabla F(x)-\nabla g(x)\hat{y})\|^2+\|\mathcal{P}_{\mathcal{X}}(x-\nabla F(x)-\nabla g(x)\hat{y})-x^*\|^2\\ &+2\langle x-\mathcal{P}_{\mathcal{X}}(x-\nabla F(x)-\nabla g(x)\hat{y}),\mathcal{P}_{\mathcal{X}}(x-\nabla F(x)-\nabla g(x)\hat{y})-x^*\rangle,\end{aligned} \tag{11}$$

then one holds

$$\begin{aligned}\dot{V} =& -\langle x-x^*,\nabla F(x)+\nabla g(x)\hat{y}-\nabla F(x^*)-\nabla g(x^*)\hat{y}^*\rangle\\ &-\|x-\mathcal{P}_{\mathcal{X}}(x-\nabla F(x)-\nabla g(x)\hat{y})\|^2-\langle x^*-\mathcal{P}_{\mathcal{X}}(x-\nabla F(x)-\nabla g(x)\hat{y}),\\ &\quad \mathcal{P}_{\mathcal{X}}(x-\nabla F(x)-\nabla g(x)\hat{y})-x+\nabla F(x)+\nabla g(x)\hat{y}\rangle\\ &-\langle\mathcal{P}_{\mathcal{X}}(x-\nabla F(x)-\nabla g(x)\hat{y})-x^*,\nabla F(x^*)+\nabla g(x^*)\hat{y}^*\rangle.\end{aligned} \tag{12}$$

From the property of projection operators and x^* is the optimal solution to the problem (3), we can get

$$\langle x^*-\mathcal{P}_{\mathcal{X}}(x-\nabla F(x)-\nabla g(x)\hat{y}),\mathcal{P}_{\mathcal{X}}(x-\nabla F(x)-\nabla g(x)\hat{y})-x+\nabla F(x)+\nabla g(x)\hat{y}\rangle \geqslant 0.$$

Since $F(x)$ and $g(x)$ are convex, then $\langle \nabla F(x)-\nabla F(x^*), x-x^*\rangle \geqslant 0$ and $\langle\mathcal{P}_{\mathcal{X}}(x-\nabla F(x)-\nabla g(x)\hat{y})-x^*,\nabla F(x^*)+\nabla g(x^*)\hat{y}^*\rangle \geqslant 0$. Thus,

$$\begin{aligned}&\langle x-x^*,\nabla F(x)+\nabla g(x)\hat{y}-\nabla F(x^*)-\nabla g(x^*)\hat{y}^*\rangle\\ \geqslant&\langle x-x^*,\nabla g(x)\hat{y}-\nabla g(x^*)\hat{y}^*\rangle\\ =&\langle x-x^*,\nabla g(x)\hat{y}-\nabla g(x^*)y^*\rangle.\end{aligned} \tag{13}$$

Furthermore, from the definition of $\hat{y}$, it has $y-\hat{y}=[y+g(x)]^- - g(x)$. Due to $\hat{y}[y+g(x)]^-=0$, $\hat{y}g(x^*)\leqslant 0$, $\hat{y}^*[y+g(x)]^-\leqslant 0$, and $\hat{y}^*g(x^*)=0$, then

$$\begin{aligned}&-\langle x-x^*,\nabla g(x)\hat{y}-\nabla g(x^*)y^*\rangle-\langle y-\hat{y},\hat{y}-y^*\rangle\\ =&-(\nabla g(x)\hat{y})^{\mathrm{T}}(x-x^*)+(\nabla g(x^*)y^*)^{\mathrm{T}}(x-x^*)\\ &-\hat{y}([y+g(x)]^- - g(x))+\hat{y}^*([y+g(x)]^- - g(x))\\ =&-\hat{y}([y+g(x)]^- - g(x)+\nabla g(x)^{\mathrm{T}}(x-x^*))\\ &+y^*([y+g(x)]^- - g(x)+\nabla g(x^*)^{\mathrm{T}}(x-x^*))\leqslant 0.\end{aligned} \tag{14}$$

Therefore, by combining the above inequalities, it has

$$\dot{V}\leqslant -\|x-\mathcal{P}_{\mathcal{X}}(x-\nabla F(x)-\nabla g(x)\hat{y})\|^2-\|\hat{y}-y\|^2=-\|\dot{x}\|^2-2\|\dot{y}\|^2\leqslant 0. \tag{15}$$

Next, we show that there is an increasing sequence $\{t_k\}$ such that

$$\lim_{k\to\infty}\|\dot{x}(t_k)\|^2+2\|\dot{y}(t_k)\|^2=0.$$

If not, there exists $r>0$ satisfying $\liminf_{t\to\infty}\|\dot{x}(t_k)\|^2+2\|\dot{y}(t_k)\|^2=r$, which means that there is $T>T_3$ such that $\|\dot{x}(t_k)\|^2+2\|\dot{y}(t_k)\|^2\geqslant\frac{r}{2}$ for all $t\geqslant T$. As a result, it can obtain that $\dot{V}(x,y)\leqslant -\frac{r}{2}$, for all $t\geqslant T$. Integrating the formula, one can obtain that $V(x(t), y(t)) \leqslant V(x(T), y(T)) - \frac{r}{2}(t-T)$. Then we have $\lim_{t\to\infty} V(x(t), y(t)) = -\infty$. This contradicts the fact that $V \geqslant 0$. Thus, we have

$$\lim_{k\to\infty} \|x(t_k) - \mathcal{P}_{\mathcal{X}}(x(t_k) - \nabla F(x(t_k)) - \nabla g(x(t_k))\hat{y}(t_k))\| = 0,$$

$$\lim_{k\to\infty} \|\hat{y}(t_k) - y(t_k)\| = 0.$$

According to (15) and $V(x, y) \geqslant \frac{1}{2}\|x - x^*\|^2 + \|y - y^*\|^2$, it deduces that $x(t)$ and $y(t)$ are bounded. In addition, from Lemma 1, there is a convergent subsequence (still denoted) $\{t_k\}$, and there are $(x_0, y_0) \in \Theta$ such that $\lim_{k\to\infty} x(t_k) = x_0$ and $\lim_{k\to\infty} y(t_k) = y_0$. Letting $x^* = x_0$ and $y^* = y_0$, similar to above analysis, we have

$$\lim_{t\to\infty} V(x(t), y(t)) = V(x^*, y^*) = 0,$$

which follows $\lim_{t\to\infty} \|x(t) - x^*\| = 0$ and $\lim_{t\to\infty} \|y(t) - y^*\| = 0$. Hence, the trajectories $x_i(t)$ $(i \in \mathcal{V})$ of the neurodynamic approach (6) asymptotically converge to the optimal solution to the problem (3). According to Theorem 1, the conclusion is obviously valid.

4 Simulation Studies

In this section, we display the effectiveness of the neurodynamic approach (6) with a twelve-agent system for the optimal allocation problem (2). The communication network of the multi-agent system is an undirected connected graph.

Table 1. Variables of the optimal allocation problem (2).

agent i	1	2	3	4	5	6	7	8	9	10	11	12
α_i	0.4	1.2	3.2	2.8	0.4	0.4	0.4	0	1.6	2.8	2.8	2
β_i	0.3	1.8	0.3	0.3	0.3	0.3	0.6	0.6	0.9	0.9	0.6	0.9
γ_i	1.8	2.1	1	3.6	2.6	2.2	3	4	1	4	3.7	2
a_i	0.1	0.24	0.07	0.06	0.8	0.14	0.02	0.11	0.08	0.2	0.04	0.06
b_i	-0.05	-0.05	-0.03	-0.01	-0.23	-0.12	-0.11	-0.07	-0.07	-0.1	-0.1	-0.2
x_i^L	0	0	0	0	0	0	0	0	0	0	0	0
x_i^U	1	1	1	2	2	2	3	3	3	4	4	4

To achieve the optimal allocation problem (2) with $D = 5.5$ and the parameters in Table 1 in this simulation, we apply the neurodynamic approach (6) by taking $k_1 = 1500$, $k_2 = 1250$, $\mu = 0.5$, and $\nu = 2$. The trajectories of $x_i(t)$ $(i = 1, 2, \cdots, 12)$ are shown in Fig. 1. Visibly, Fig. 1 illustrates that the neurodynamic approach (6) ia able to get the optimal solution

$$x^* = [0.85, 0, 0.13, 0.14, 0.42, 0.85, 0.76, 2.42, 0.09, 0.05, 0.11, 0.11]$$

of the problem (3) and shows the convergence of the neurodynamic approach (6). It can be seen from Fig. 2 that the states $y_i(t)$ and $z_i(t)$ reach consensus within a finite time, which is consistent with the theoretical results in this paper. Furthermore, Fig. 3(a) describes the evaluations of the total cost value, and Fig. 3(b) shows that the global equation constraint of the problem (3) with resource losses can be satisfied. To sum up, these numerical results verify that the neurodynamic approach designed in this paper is effective for solving the optimal allocation problem with resource losses.

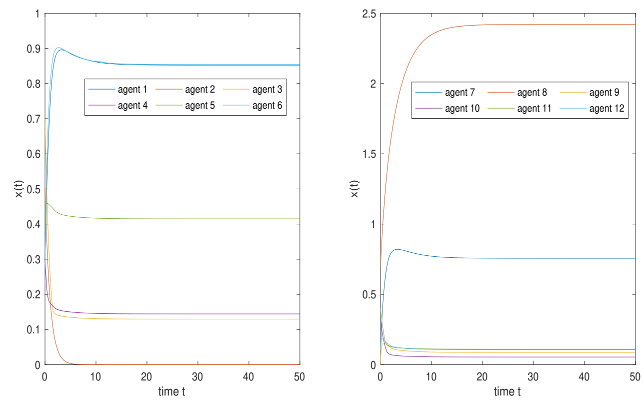


Fig. 1. The trajectories of $x_i(t)$ generated by neurodynamic approach (6).

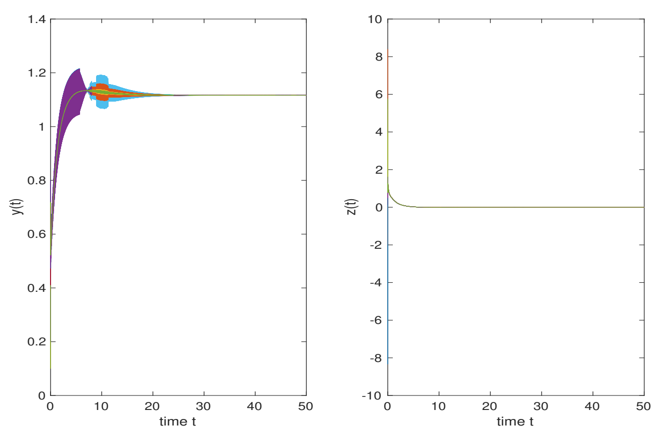
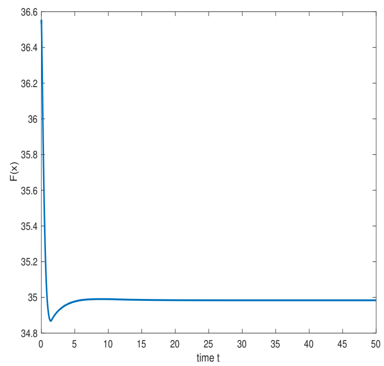
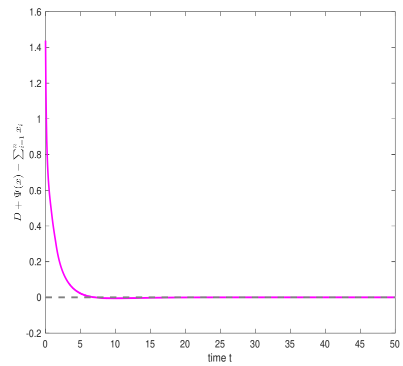


Fig. 2. The trajectories of $y_i(t)$ and $z_i(t)$ generated by neurodynamic approach (6).



(a) Evolution of $F(x)$



(b) Evolution of $D + \Psi(x) - \sum_{i=1}^{n} x_i$

Fig. 3. The total cost and global constraint with resource losses in the optimal allocation problem (2).

5 Conclusion

In this paper, we investigated the distributed optimal allocation problem with separable resource losses. Through applying finite-time tracking technology and the properties of the projection operator, a distributed neurodynamic approach based on multi-agent system was designed and analyzed. Moreover, we showed that the states of the proposed neurodynamic approach can converge to the optimal solution of the considered problem both theoretically and numerically.

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