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ALGORITHM TO COMPUTE THE GLOBAL SERVICE TIME DISTRIBUTION IN AN OPEN NETWORK OF $M|G|\infty$ QUEUES THROUGH LAPLACE TRANSFORMS

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ABSTRACT

The Laplace transform is a widely used tool in the study of probability distributions, often allowing for a simpler determination of the p.d.f.'s and d.f.'s, and having the possibility to work as a "moments generating function". In this paper it is considered a situation not so simple, as it is the case of the $M|G|\infty$ *queue busy period length distribution. Attention will also be given the respective tail Laplace transform. Then, in the context of an open queues network, which nodes behave as* $M|G|\infty$ *queues, the Laplace transform will be used to construct an algorithm to determine the Laplace transform of the global service time length of a customer during their stay on the network distribution.*

Keywords: Laplace transform, $M|G|\infty$, busy period, queues network, algorithm.

***Mathematics Subject Classification*:** 44A10 and 60G99

INTRODUCTION

In the $M|G|\infty$ queue, customers arrive according to a Poisson process at rate λ. Upon its arrival receive immediately a service with time length $d.\,f.$ $G(\cdot)$ and mean α. The traffic intensity is $\rho=\lambda\alpha$, see for instance [7].

A **network of queues** is a collection of nodes, arbitrarily connected by arcs, across which the customers travel instantaneously and

- There is an arrival process associated to each node,

- There is a **commutation process** which commands the various costumers' paths,

The arrival processes may be composed of **exogenous arrivals**, from the outside of the collection, and of **endogenous arrivals**, from the other collection nodes. Call

$$\Lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_J \end{bmatrix} \quad (1.1)$$

the network exogenous arrival rates vector, where λ_j is the exogenous arrival rate at node j and $\lambda = \sum_{j=1}^{J} \lambda_j$.

A network is **open** if any customer may enter or leave it. A network is **closed** if it has a fixed number of customers that travel from node to node and there are neither arrivals from the outside of the collection nor departures. A network open for some customers and closed for others is said **mixed**.

The commutation process rules, for each costumer that abandons a node, which node it can visit then or if it leaves the network. In a network with *J* nodes, the matrix

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1j} \\ p_{21} & p_{22} & \cdots & p_{2j} \\ \vdots & \vdots & \vdots & \vdots \\ p_{j1} & p_{j2} & \cdots & p_{jj} \end{bmatrix} (1.2)$$

is the commutation process matrix, being p_{jl} the probability of a customer, after ending its service at node j, go to node l, $j, l = 1,2, \dots J$. The probability $q_j = 1 - \sum_{l=1}^{J} p_{jl}$ is the probability that a customer leaves the network from node $j, j = 1,2, \dots J$. Call now γ_j, the total – from the outside of the network and from the other nodes – customers arrival rate at node *j* and

$$\Gamma = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_J \end{bmatrix} (1.3)$$

the network exogenous arrival rates vector. If the network is stable, the following equality – **traffic equations** – holds:

$$\Gamma^T = \Lambda^T + \Gamma^T \mathrm{P} \quad (1.4).$$

Note that they may be written as $\Gamma^T = \Lambda^T (I - P)^{-1}$.

For more details on networks of queues see [2] and [15].

In next section, for open networks of queues, which nodes are $M|G|\infty$ queues, will be constructed an algorithm to determine the Laplace transform of the distribution of the global service time length of a client during their stay on the network, see [11]. This work finishes with the presentation of a conclusions section and a short list of references.

ALGORITHM TO COMPUTE THE GLOBAL SERVICE TIME DISTRIBUTION IN AN OPEN NETWORK OF $M|G|\infty$ QUEUES THROUGH LAPLACE TRANSFORMS

An open network of queues with infinite servers in each node, with Poisson process exogenous arrivals, may be looked like a $M|G|\infty$ queue. The service time is the sojourn time of a customer in the network, see [13, 14].

Note that the sojourn time is the mixture of the sums of the services corresponding to each path that a customer may have in the network. The total time spent in a path by a customer distribution is so the convolution of the service time distributions in each node belonging to the path, since those service times are independent. Each one of these convolutions is a parcel in the mixture which weight is given by the path probability. Each path starts in a node *j* with probability $\frac{\lambda_j}{\lambda}$ and ends in node *k* with probability $1-\sum_{j=1}^{J} p_{kj}$.

As the Laplace transform of a convolution of two functions is the product the two those functions Laplace transforms and having in mind the traffic equations seen above (expression (1.4)):

- Denote S the sojourn time of a costumer in the network and S_j its service time at node $j, j = 1,2, \dots J$. Be $G(t)$ and $G_j(t)$ the S and S_j distribution functions, respectively, and $\bar{G}(s)$ and $\bar{G}_j(s)$ the Laplace transforms,
- Define

$$\Lambda(s) = \begin{bmatrix} \lambda_1 \bar{G}_1(s) \\ \lambda_2 \bar{G}_2(s) \\ \vdots \\ \lambda_J \bar{G}_J(s) \end{bmatrix} \text{ and } P(s) = \begin{bmatrix} p_{11}\bar{G}_1(s) & p_{12}\bar{G}_2(s) & \dots & p_{1J}\bar{G}_J(s) \\ p_{21}\bar{G}_1(s) & p_{22}\bar{G}_2(s) & \dots & p_{2J}\bar{G}_J(s) \\ \vdots & \vdots & & \vdots \\ p_{J1}\bar{G}_1(s) & p_{J2}\bar{G}_2(s) & & p_{JJ}\bar{G}_J(s) \end{bmatrix} \quad (2.1)$$

- It results

$$\bar{G}(s) = \sum_{n=0}^{\infty} (\lambda^{-1}\Lambda^T(s)P^n(s)(I-P)A) \qquad (2.2),$$

- And finally, using the Leontief's matrix properties, the global service time distribution Laplace transform service time for a customer during its permanence in the network is given by

$$\bar{G}(s) = \lambda^{-1}\Lambda^T(s)\big(I-P(s)\big)^{-1}(I-P)A \qquad (2.3),$$

where *I* is the identity matrix with the same order as *P* and *A* is a column with J 1`s, for the Laplace Transform service time, confer with [11].

So, the problem in terms of Laplace transforms is operationally simple. The "problems" arrive when inverting the Laplace transform. As usual the situation is not bad when using exponential expressions.

Let's see some examples:

- For $s = 0$, expression (2.3) becomes

$$\bar{G}(0) = \lambda^{-1} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_J \end{bmatrix}^T (I - P)^{-1}(I - P) \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = 1,$$

as expected,

- For $J = 1$, expression (2.3) becomes

$$\bar{G}(s) = \lambda^{-1}\lambda\, \bar{G}_1(s)\left(1 - p_{11}\bar{G}_1(s)\right)^{-1}(1 - p_{11}) = \frac{(1 - p_{11})\bar{G}_1(s)}{1 - p_{11}\bar{G}_1(s)}$$

that for $p_{11} = 0$ becomes $\bar{G}(s) = \bar{G}_1(s)$, and for $p_{11} = 1$ becomes $\bar{G}(s) = 0$, as expected

-Suppose the following network of infinite servers' queues, schematized in Fig. 1,

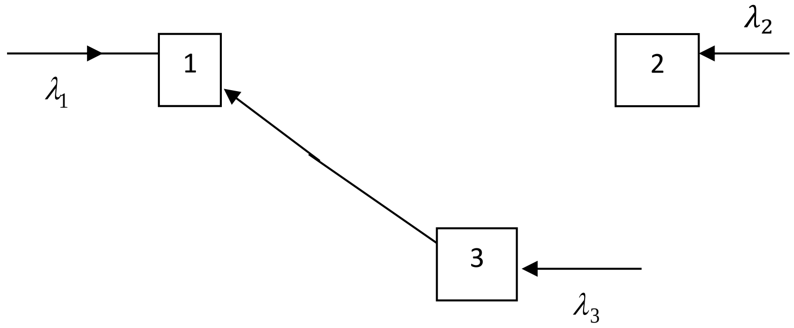

Fig. 1. Infinite servers' queues network scheme

$$\Lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} (1-q)\lambda \\ (1-p)q\lambda \\ pq\lambda \end{bmatrix}, 0 \leq p, q \leq 1 \text{ and obviously } P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \text{ So,}$$

$\Lambda(s) = \begin{bmatrix} (1-q)\lambda\bar{G}_1(s) \\ (1-p)q\lambda\bar{G}_2(s) \\ pq\lambda\bar{G}_3(s) \end{bmatrix}$, $P(s) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ p_{31}\bar{G}_1(s) & 0 & 0 \end{bmatrix}$, being the system, globally, a $M|G|\infty$ queue and

$$\overline{G}(s) = \left[(1-q)\overline{G}_1(s) + (1-p)q\overline{G}_2(s) + pq\overline{G}_1(s)\overline{G}_3(s)\right].$$

So, the service time distribution function is

$$G(t) = (1-q)G_1(t) + (1-p)qG_2(t) + pqG_{13}(t).$$

The symbolG_{13} represents the distribution function of the convolution of the service time distributions in nodes 1 and 3.

The following situation can be modelled by this network:

- Suppose a fleet which failures repairs occur in a base or in a remote station. The whole failures detected in the base are there repaired. Failures detected in the station are repaired in the base with probability p , being necessary to carry them to the base, and the others in the station. Here the service time is the time that goes from the instant at which the failure occurs till the one at which it is completely repaired. When it is necessary to carry an item with a failure from the remote station to the base it is assumed that it is immediately possible, being the service time, now, the time that the transportation lasts. It is also supposed that the failures occur according to a Poisson process at rate λ , being detected in the remote station with probability q and the others in the base. This situation may be modeled as a network of queues with three nodes in Fig.1, at which 1 is the base, 2 is the remote station and 3 considers the required transports from the remote station to the base. The model makes possible to calculate system performance measures.

CONCLUSIONS

In this text the operational qualities and also the research incentive of the Laplace transform are well known. The results presented are of both types: purely quantitative and of theoretical scope.

In studies about stochastic processes, of which queues are part, it is very common to use this tool.

The situation is like that which occurs in differential equations: great simplification in operational matters, not always accompanied by comparable simplification when it is necessary to reverse the transform.

It happens that in the case of stochastic processes, it is often possible to collect the fruits of the research without recourse to inversion.

From the presented results it is worth noting the formula (4.3) for its simplicity and evident utility, where the qualities of the Laplace transform are quite explored.

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