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Parametric Chu Translation

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Bellin [1] translates multiplicative-additive linear logic to its intuitionistic fragment by relating the Chu construction to trips on a proof net, seemingly posing an alternative to negative translation. However, his translation is not sound in the sense that not all valid intuitionistic sequents in its image correspond to valid classical ones. By directly analyzing two-sided classical sequents, we develop a sound generalization thereof inspired by parametric negative translation [2, 8] that also handles the exponentials.

1 Introduction

Notation: A, B, and p are formulas, $\neg$ is linear negation, Γ and Δ are contexts (multisets of formulas), and δ is a stoup with at most one formula. Intuitionistic objects are blue, classical ones are red, and undistinguished ones are black. Refer to [5] for the two-sided sequent calculus for classical linear logic and [4] for its intuitionistic restriction.

Correspondence between classical and intuitionistic linear logic can be established via *parametric negative translation* [2, 8]. Defining *parametric negation* $\neg_p A \triangleq A \multimap p$, let $[\![A]\!]_p$ send classical formulas to intuitionistic ones by applying double negations as needed. For example, $[\![A \parr B]\!]_p \triangleq \neg_p(\neg_p[\![A]\!]_p \otimes \neg_p[\![B]\!]_p)$. One must then establish the following metatheorems [2].

$$\Gamma \vdash \Delta \underset{\text{soundness}}{\overset{\text{preservation}}{\rightleftarrows}} [\![\Gamma]\!]_p, \neg_p[\![\Delta]\!]_p \vdash p \text{ for all } p$$

In other words, a classical proof corresponds to an intuitionistic proof-by-contradiction relative to p. However, the (double) negations introduced by the translation obscure the structure of the original classical proof, raising the question: is there a translation that closely preserves classical proof structure? Bellin's [1] translation, which relates the Chu construction [3] to the sequent calculus by way of trips on a proof net, is one such candidate: the translation is split into input and output parts A_I and A_O, respectively, such that:

$$\vdash \Gamma, C \xrightarrow{\text{preservation}} \Gamma_I \vdash C_O$$

In particular, A_O sends the intuitionistic connectives to themselves and $A_I \triangleq (A^*)_O$ where A^* is the de Morgan dual of A. Then, $\bot_O \triangleq \top$ and $(A \parr B)_O \triangleq (A_I \multimap B_O) \,\&\, (B_I \multimap A_O)$, making intuitionistic sense of $\parr$ with a different proof theory than above and in full intuitionistic linear logic [6]. However, this translation is not sound: $\vdash \bot_O$, i.e., $\vdash \top$ holds, but $\vdash \bot$ does not. Moreover, it does not cover the exponentials.

By directly analyzing two-sided classical sequents, we present a parametric generalization of Bellin's translation in figure 1 satisfying the metatheorems below that also handles the exponentials.

$$\Gamma \vdash \delta \underset{\text{soundness}}{\overset{\text{preservation}}{\rightleftarrows}} [\![\Gamma]\!]_p \vdash [\delta]_p \text{ for all } p$$

The remainder of this document develops the translation and its metatheory incrementally.

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$$[\cdot]_p \triangleq p \quad [A]_p \triangleq \llbracket A \rrbracket_p \quad \rrbracket A \llbracket_p \triangleq \llbracket A^* \rrbracket_p \quad \llbracket \neg A \rrbracket \triangleq \rrbracket A \llbracket_p$$
$$\llbracket \mathbf{1} \rrbracket_p \triangleq \mathbf{1} \quad \llbracket A \otimes B \rrbracket_p \triangleq \llbracket A \rrbracket_p \otimes \llbracket B \rrbracket_p \quad \llbracket \bot \rrbracket_p \triangleq p \quad \llbracket A \parr B \rrbracket_p \triangleq (\rrbracket A \llbracket_p \multimap \llbracket B \rrbracket_p) \,\&\, (\rrbracket B \llbracket_p \multimap \llbracket A \rrbracket_p)$$
$$\llbracket \mathbf{0} \rrbracket_p \triangleq \mathbf{0} \quad \llbracket A \oplus B \rrbracket_p \triangleq \llbracket A \rrbracket_p \oplus \llbracket B \rrbracket_p \quad \llbracket \top \rrbracket_p \triangleq \top \quad \llbracket A \,\&\, B \rrbracket_p \triangleq \llbracket A \rrbracket_p \,\&\, \llbracket B \rrbracket_p$$
$$\llbracket !A \rrbracket_p \triangleq !\llbracket A \rrbracket_p \quad \llbracket ?A \rrbracket_p \triangleq \neg_p ! \rrbracket A \llbracket_p$$

Figure 1: Parametric Chu Translation

2 Multiplicatives and Additives

To develop our translation, let us consider a cut- and initial-free derivation D of the classical sequent $\Gamma \vdash \delta$—when δ is empty, D derives a contradiction from the hypotheses in Γ. Otherwise, it is a direct proof of the formula in δ. The key to our translation $\llbracket A \rrbracket_p$, then, is to convert the former to a proof-by-contradiction relative to the parameter p and the latter to a direct intuitionistic proof. Formally, we want to prove preservation—derive $\llbracket \Gamma \rrbracket_p \vdash [\delta]_p$ where $[\cdot]_p \triangleq p$ and $[A]_p \triangleq \llbracket A \rrbracket_p$ by induction on D. Let us consider the cases for the classical multiplicative-additive connectives, i.e., when D ends in $\bot$L, $\bot$R, $\neg$L, $\neg$R, $\parr$L, or $\parr$R. While the restriction to at most one succedent seems arbitrary, it is essential to making sense of $\parr$.

Since p is the intuitionistic target for contradiction, let $\llbracket \bot \rrbracket_p \triangleq p$. Then, the first two cases are straightforward. Below, IH stands for the inductive hypothesis and the dashed lines indicate admissible rules.

$$D = \dfrac{}{\bot \vdash \cdot}\ \bot\text{L} \implies \dfrac{}{p \vdash p}\ \text{init} \qquad\qquad D = \dfrac{\begin{matrix} D' \\ \vdots \\ \Gamma \vdash \cdot \end{matrix}}{\Gamma \vdash \bot}\ \bot\text{R} \implies \dfrac{\begin{matrix} D' \\ \vdots \\ \Gamma \vdash \cdot \end{matrix}}{\llbracket \Gamma \rrbracket_p \vdash p}\ \text{IH}$$

Now, it is tempting to let $\llbracket \neg A \rrbracket_p = \neg_p \llbracket A \rrbracket_p$. Although preservation of $\neg$R succeeds, $\neg$L does not, since, by design, our inductive hypothesis does not apply to sequents with more than one succedent.

$$D = \dfrac{\begin{matrix} D' \\ \vdots \\ \Gamma, A \vdash \cdot \end{matrix}}{\Gamma \vdash \neg A}\ \neg\text{R} \implies \dfrac{\dfrac{\begin{matrix} D' \\ \vdots \\ \Gamma, A \vdash \cdot \end{matrix}}{\llbracket \Gamma \rrbracket_p, \llbracket A \rrbracket_p \vdash p}\ \text{IH}}{\llbracket \Gamma \rrbracket_p \vdash \neg_p \llbracket A \rrbracket_p}\ \multimap\text{R} \qquad\qquad D = \dfrac{\begin{matrix} D' \\ \vdots \\ \Gamma \vdash A, \delta \end{matrix}}{\Gamma, \neg A \vdash \delta}\ \neg\text{L} \implies ?$$

Instead, we consider de Morgan duality; recall the following standard lemma.

Lemma 1 (de Morgan duality)**.**

1. If $\overset{D}{\vdots}$ $\Gamma, A \vdash \Delta$, then there exists D' such that $\overset{D'}{\vdots}$ $\Gamma \vdash A^*, \Delta$.
2. If $\overset{D}{\vdots}$ $\Gamma \vdash A, \Delta$, then there exists D' such that $\overset{D'}{\vdots}$ $\Gamma, A^* \vdash \Delta$.

Moreover, D and D' have the same height.

Thus, let $\llbracket \neg A \rrbracket_p \triangleq \rrbracket A \llbracket_p$ where $\rrbracket A \llbracket_p \triangleq \llbracket A^* \rrbracket_p$. Then, we can complete preservation of classical negation. Note that the uses of IH are well-defined since lemma 1 is height-preserving.

$$D = \dfrac{\overset{D'}{\vdots}\atop \Gamma, A \vdash \cdot}{\Gamma \vdash \neg A}\ \neg\text{R} \implies \dfrac{\dfrac{\overset{D'}{\vdots}\atop \Gamma, A \vdash \cdot}{\Gamma \vdash A^*}\ \text{p. 1}}{\llbracket \Gamma \rrbracket_p \vdash \llbracket A^* \rrbracket_p}\ \text{IH} \qquad D = \dfrac{\overset{D}{\vdots}\atop \Gamma \vdash A, \delta}{\Gamma, \neg A \vdash \delta}\ \neg\text{L} \implies \dfrac{\dfrac{\overset{D'}{\vdots}\atop \Gamma \vdash A, \delta}{\Gamma, A^* \vdash \delta}\ \text{p. 2}}{\llbracket \Gamma \rrbracket_p, \llbracket A^* \rrbracket_p \vdash \delta}\ \text{IH}$$

Now, the case for $\parr L$ splits into two sub-cases depending on which sub-derivation proves δ.

$$D = \dfrac{\overset{D_1}{\vdots}\atop \Gamma, A \vdash \cdot \qquad \overset{D_2}{\vdots}\atop B, \Delta \vdash \delta}{\Gamma, A \parr B, \Delta \vdash \delta}\ \parr\text{L} \qquad D = \dfrac{\overset{D'_1}{\vdots}\atop \Gamma, A \vdash \delta \qquad \overset{D'_2}{\vdots}\atop B, \Delta \vdash \cdot}{\Gamma, A \parr B, \Delta \vdash \delta}\ \parr\text{L}$$

The restriction to at most one succedent pays off: it reveals that each sub-case can be viewed as an application of $\multimap$L.

$$E = \dfrac{\dfrac{\dfrac{\overset{D_1}{\vdots}\atop \Gamma, A \vdash \cdot}{\Gamma \vdash A^*}\ \text{p. 1}}{\llbracket \Gamma \rrbracket_p \vdash \llbracket A^* \rrbracket_p}\ \text{IH} \qquad \dfrac{\overset{D_2}{\vdots}\atop B, \Delta \vdash \delta}{\llbracket B \rrbracket_p, \llbracket \Delta \rrbracket_p \vdash [\delta]_p}\ \text{IH}}{\llbracket \Gamma \rrbracket_p, \rrbracket A \llbracket_p \multimap \llbracket B \rrbracket_p, \llbracket \Delta \rrbracket_p \vdash [\delta]_p}\ \multimap\text{L} \qquad F = \dfrac{\dfrac{\dfrac{\overset{D'_2}{\vdots}\atop B, \Delta \vdash \cdot}{\Delta \vdash B^*}\ \text{p. 1}}{\llbracket \Delta \rrbracket_p \vdash \llbracket B^* \rrbracket_p}\ \text{IH} \qquad \dfrac{\overset{D'_1}{\vdots}\atop \Gamma, A \vdash \delta}{\llbracket A \rrbracket_p, \llbracket \Gamma \rrbracket_p \vdash [\delta]_p}\ \text{IH}}{\llbracket \Gamma \rrbracket_p, \rrbracket B \llbracket_p \multimap \llbracket A \rrbracket_p, \llbracket \Delta \rrbracket_p \vdash [\delta]_p}\ \multimap\text{L}$$

Thus, letting $\llbracket A \parr B \rrbracket_p \triangleq (\rrbracket A \llbracket_p \multimap \llbracket B \rrbracket_p) \,\&\, (\rrbracket B \llbracket_p \multimap \llbracket A \rrbracket_p)$, we can complete preservation for this case as follows.

$$\dfrac{\overset{E}{\vdots}\atop \llbracket \Gamma \rrbracket_p, \rrbracket A \llbracket_p \multimap \llbracket B \rrbracket_p, \llbracket \Delta \rrbracket_p \vdash [\delta]_p}{\llbracket \Gamma \rrbracket_p, \llbracket A \parr B \rrbracket_p, \llbracket \Delta \rrbracket_p \vdash [\delta]_p}\ \&\text{L1} \qquad \dfrac{\overset{F}{\vdots}\atop \llbracket \Gamma \rrbracket_p, \rrbracket B \llbracket_p \multimap \llbracket A \rrbracket_p, \llbracket \Delta \rrbracket_p \vdash [\delta]_p}{\llbracket \Gamma \rrbracket_p, \llbracket A \parr B \rrbracket_p, \llbracket \Delta \rrbracket_p \vdash [\delta]_p}\ \&\text{L2}$$

Once again, the use of de Morgan duality in lieu of parametric negation allows us to complete the case when D ends in $\parr$R.

$$D = \dfrac{\overset{D'}{\vdots}\atop \Gamma \vdash A, B}{\Gamma \vdash A \parr B}\ \parr\text{R} \implies \dfrac{\dfrac{\dfrac{\dfrac{\overset{D'}{\vdots}\atop \Gamma \vdash A, B}{\Gamma, A^* \vdash B}\ \text{p. 2}}{\llbracket \Gamma \rrbracket_p, \llbracket A^* \rrbracket_p \vdash \llbracket B \rrbracket_p}\ \text{IH}}{\llbracket \Gamma \rrbracket_p \vdash \rrbracket A \llbracket_p \multimap \llbracket B \rrbracket_p}\ \multimap\text{R} \qquad \dfrac{\dfrac{\dfrac{\overset{D'}{\vdots}\atop \Gamma \vdash A, B}{\Gamma, B^* \vdash A}\ \text{p. 2}}{\llbracket \Gamma \rrbracket_p, \llbracket B^* \rrbracket_p \vdash \llbracket A \rrbracket_p}\ \text{IH}}{\llbracket \Gamma \rrbracket \vdash \rrbracket B \llbracket_p \multimap \llbracket A \rrbracket_p}\ \multimap\text{R}}{\llbracket \Gamma \rrbracket_p \vdash (\rrbracket A \llbracket_p \multimap \llbracket B \rrbracket_p) \,\&\, (\rrbracket B \llbracket_p \multimap \llbracket A \rrbracket_p)}\ \&\text{R}$$

We have completed our translation guided by the proof of preservation (theorem statement below).

Theorem 1 (Preservation)**.** *If $\Gamma \vdash \delta$, then $\llbracket \Gamma \rrbracket_p \vdash [\delta]_p$ for all p.*

Bringing our discussion back to the Chu construction [3], $[\![A]\!]_p$ and $]\!]A[\![_p$ generalize A_O and A_I, respectively. Bellin's [1] translation determines a functor from a free categorical model of classical linear logic to a special case of the Chu construction with $\top$ as the dualizing object. Thus, $\bot$ is sent to $\top$, so the translation is not sound. Like Bellin, we do not know whether the corresponding functor is faithful becaause that requires investigating the identity of proofs, which may be better served by an alternate proof representation like proof nets or focused derivations. However, the functor is certainly not full: an anonymous referee indicated that $(\bot \,\&\, \bot) \parr (\mathbf{1} \oplus \mathbf{1})$ has more (intuitionistic) proofs in the image of the translation than classical ones. Thus, the categorical semantics of this translation requires further development.

To show soundness, like Chang et al. [2], we observe that when $p = \bot$, the translation is essentially the identity with respect to classical equivalence.

Lemma 2 (Pre-soundness)**.** Let $\equiv$ be classical equivalence. Then, $[\![A]\!]_\bot \equiv A$.

Proof. By induction on A, we only show the interesting cases.

1. If $A = \bot$, then $[\![\bot]\!]_\bot \equiv \bot$ by definition.
2. If $A = \neg A_1$, then $[\![\neg A_1]\!]_\bot =]\!]A_1[\![_\bot = [\![A_1^*]\!]_\bot \equiv A_1^* \equiv \neg A_1$ by the IH and equivalence of $\neg$ and $*$.
3. If $A = A_1 \parr A_2$, then:

$$\begin{aligned}
[\![A_1 \parr A_2]\!]_\bot &= (]\!]A_1[\![_\bot \multimap [\![A_2]\!]_\bot) \,\&\, (]\!]A_2[\![_\bot \multimap [\![A_1]\!]_\bot) \\
&\equiv (A_1^* \multimap A_2) \,\&\, (A_2^* \multimap A_1) && \text{by IH} \\
&\equiv (\neg A_1^* \parr A_2) \,\&\, (\neg A_2^* \parr A_1) \\
&\equiv (A_1 \parr A_2) \,\&\, (A_2 \parr A_1) && \text{by involutivity of negation} \\
&\equiv A_1 \parr A_2 && \text{by additive idempotence}
\end{aligned}$$

□

Finally, soundness at the level of proofs is a corollary, observing that intuitionistic proofs are also classical ones.

Theorem 2 (Soundness)**.** *If $[\![\Gamma]\!]_p \vdash [\delta]_p$ for all p, then $\Gamma \vdash \delta$.*

Proof. Setting $p = \bot$, we have $\Gamma \vdash [\delta]_\bot$ by lemma 2. If δ is empty, then we have $\Gamma \vdash \bot$, i.e., $\Gamma \vdash \bot$ by lemma 2. Since $\bot$R is invertible, we have $\Gamma \vdash \cdot$. Otherwise, if $\delta = A$, then $\Gamma \vdash [\![A]\!]_\bot$, i.e., $\Gamma \vdash A$ by lemma 2. □

3 Exponentials

It remains to translate $!A$ and $?A$. The former is straightforward: $[\![!A]\!]_p \triangleq\, ![\![A]\!]_p$. Since we cannot count on the presence of $?$, we utilize double negation for the first time: $[\![?A]\!]_p \triangleq \neg_p !]\!]A[\![_p$. While this is similar to constructing a coexponential by double duality in a special case of the Chu construction [7], our notion of double negation is staged across the intuitionistic and classical layers via $\neg_p$ and $*$, respectively. Preservation for derivations D ending in $?$L (promotion), $?$R (dereliction), wR (weakening), and cR (contraction) reveals an interesting interplay between both forms of negation. Now, the first case splits into two sub-cases depending on whether δ is empty or not.

$$D = \dfrac{\begin{matrix} D' \\ \vdots \\ !\Gamma, A \vdash \cdot \end{matrix}}{!\Gamma, ?A \vdash \cdot}\ ?\mathrm{L} \quad \Longrightarrow \quad \dfrac{\dfrac{\begin{matrix} D' \\ \vdots \\ \dfrac{!\Gamma, A \vdash \cdot}{!\Gamma \vdash A^*}\ \text{p. } 1 \\ \vdots\ \text{IH} \\ ![\![\Gamma]\!]_p \vdash [\![A^*]\!]_p \end{matrix}}{![\![\Gamma]\!]_p \vdash !]\!]A[\![_p}\ !\mathrm{R} \qquad \dfrac{}{p \vdash p}\ \text{init}}{![\![\Gamma]\!]_p, \neg_p !]\!]A[\![_p \vdash p}\ \multimap\mathrm{L}$$

$$D = \dfrac{\begin{matrix} D' \\ \vdots \\ !\Gamma, A \vdash ?B \end{matrix}}{!\Gamma, ?A \vdash ?B}\ ?\mathrm{L} \quad \Longrightarrow \quad \dfrac{\dfrac{\dfrac{\begin{matrix} D' \\ \vdots \\ \dfrac{!\Gamma, ?A \vdash ?B}{!\Gamma, !B^* \vdash !A^*}\ \text{p. } 1,2 \\ \vdots\ \text{IH} \\ ![\![\Gamma]\!]_p, ![\![B^*]\!]_p \vdash [\![A^*]\!]_p \end{matrix}}{![\![\Gamma]\!]_p, !]\!]B[\![_p \vdash !]\!]A[\![_p}\ !\mathrm{R} \qquad \dfrac{}{p \vdash p}\ \text{init}}{![\![\Gamma]\!]_p, \neg_p !]\!]A[\![_p, !]\!]B[\![_p \vdash p}\ \multimap\mathrm{L}}{![\![\Gamma]\!]_p, \neg_p !]\!]A[\![_p \vdash \neg_p !]\!]B[\![_p}\ \multimap\mathrm{R}$$

The remaining cases are more straightforward.

$$D = \dfrac{\begin{matrix} D' \\ \vdots \\ \Gamma \vdash A \end{matrix}}{\Gamma \vdash ?A}\ ?\mathrm{R} \quad \Longrightarrow \quad \dfrac{\dfrac{\begin{matrix} D' \\ \vdots \\ \dfrac{\Gamma \vdash A}{\Gamma, A^* \vdash \cdot}\ \text{p. } 2 \\ \vdots\ \text{IH} \\ [\![\Gamma]\!]_p, [\![A^*]\!]_p \vdash p \end{matrix}}{[\![\Gamma]\!]_p, !]\!]A[\![_p \vdash p}\ !\mathrm{L}}{[\![\Gamma]\!]_p \vdash \neg_p !]\!]A[\![_p}\ \multimap\mathrm{R}$$

$$D = \dfrac{\begin{matrix} D' \\ \vdots \\ \Gamma \vdash \cdot \end{matrix}}{\Gamma \vdash ?A}\ \mathrm{wR} \quad \Longrightarrow \quad \dfrac{\dfrac{\begin{matrix} D' \\ \vdots \\ \Gamma \vdash \cdot \\ \vdots\ \text{IH} \\ [\![\Gamma]\!]_p \vdash p \end{matrix}}{[\![\Gamma]\!]_p, !]\!]A[\![_p \vdash p}\ \mathrm{wL}}{[\![\Gamma]\!]_p \vdash \neg_p !]\!]A[\![_p}\ \multimap\mathrm{R} \qquad D = \dfrac{\begin{matrix} D' \\ \vdots \\ \Gamma \vdash ?A, ?A \end{matrix}}{\Gamma \vdash ?A}\ \mathrm{cR} \quad \Longrightarrow \quad \dfrac{\dfrac{\begin{matrix} D' \\ \vdots \\ \dfrac{\Gamma \vdash ?A, ?A}{\Gamma, !A^*, !A^* \vdash \cdot}\ \text{p. } 2 \times 2 \\ \vdots\ \text{IH} \\ [\![\Gamma]\!]_p, ![\![A^*]\!]_p, ![\![A^*]\!]_p \vdash p \end{matrix}}{[\![\Gamma]\!]_p, ![\![A^*]\!]_p \vdash p}\ \mathrm{cL}}{[\![\Gamma]\!]_p \vdash \neg_p !]\!]A[\![_p}\ \multimap\mathrm{R}$$

Pre-soundness for both connectives is immediate, completing the metatheory.

4 Exploiting Parametricity

As a bonus, we can show preservation and soundness of translations from extensions to classical linear logic by fixing a value of p (in lieu of parametricity in p). For example, $p = \mathbf{1}$ admits nullary and binary MIX rules. Formally, we prove $\Gamma \vdash \delta$ iff $[\![\Gamma]\!]_{\mathbf{1}} \vdash [\delta]_{\mathbf{1}}$. In the forward direction, the only interesting cases involve derivations D ending in the MIX rules.

$$D = \dfrac{}{\cdot \vdash \cdot}\ \mathrm{MIX}_0 \implies \dfrac{}{\cdot \vdash \mathbf{1}}\ \mathbf{1}\mathrm{R}$$

The case of the binary MIX rule splits on the premise that proves δ.

$$D = \dfrac{\begin{matrix} D_1 \\ \vdots \\ \Gamma \vdash \cdot \end{matrix} \quad \begin{matrix} D_2 \\ \vdots \\ \Delta \vdash \delta \end{matrix}}{\Gamma, \Delta \vdash \delta}\ \mathrm{MIX}_2 \implies \dfrac{\dfrac{\begin{matrix} D_1 \\ \vdots \\ \Gamma \vdash \cdot \end{matrix}}{[\![\Gamma]\!]_{\mathbf{1}} \vdash \mathbf{1}}\ \mathrm{IH} \quad \dfrac{\dfrac{\begin{matrix} D_2 \\ \vdots \\ \Delta \vdash \delta \end{matrix}}{[\![\Delta]\!]_{\mathbf{1}} \vdash [\delta]_{\mathbf{1}}}\ \mathrm{IH}}{[\![\Delta]\!]_{\mathbf{1}}, \mathbf{1} \vdash [\delta]_{\mathbf{1}}}\ \mathbf{1}\mathrm{L}}{[\![\Gamma]\!]_{\mathbf{1}}, [\![\Delta]\!]_{\mathbf{1}} \vdash [\delta]_{\mathbf{1}}}\ \mathrm{cut}$$

$$D = \dfrac{\begin{matrix} D_1 \\ \vdots \\ \Gamma \vdash \delta \end{matrix} \quad \begin{matrix} D_2 \\ \vdots \\ \Delta \vdash \cdot \end{matrix}}{\Gamma, \Delta \vdash \delta}\ \mathrm{MIX}_2 \implies \dfrac{\dfrac{\begin{matrix} D_2 \\ \vdots \\ \Delta \vdash \cdot \end{matrix}}{[\![\Delta]\!]_{\mathbf{1}} \vdash \mathbf{1}}\ \mathrm{IH} \quad \dfrac{\dfrac{\begin{matrix} D_1 \\ \vdots \\ \Gamma \vdash \delta \end{matrix}}{[\![\Gamma]\!]_{\mathbf{1}} \vdash [\delta]_{\mathbf{1}}}\ \mathrm{IH}}{[\![\Gamma]\!]_{\mathbf{1}}, \mathbf{1} \vdash [\delta]_{\mathbf{1}}}\ \mathbf{1}\mathrm{L}}{[\![\Gamma]\!]_{\mathbf{1}}, [\![\Delta]\!]_{\mathbf{1}} \vdash [\delta]_{\mathbf{1}}}\ \mathrm{cut}$$

Once again, soundness is established by pre-soundness, which is immediate from $\mathbf{1} \equiv \bot$ in the presence of the MIX rules. For further examples, please see [2, 8].

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