Normalization and Taylor expansion of lambda-terms

Federico Olimpieri

May 24, 2018

Normalization and Taylor expansion of λ-terms

Federico Olimpieri

May 24, 2018

1 Introduction

The Taylor expansion of ordinary λ-terms has been introduced in [ER08] as a syntactic counterpart of the quantitative semantics of linear logic in a λ-calculus setting. The aim of this work is to characterize three fundamental normalization properties in λ-calculus trough the Taylor expansion. The general proof strategy consists in stating the dependence of ordinary reduction strategies on their resource counterparts and in finding a convenient resource term in the support of the Taylor expansion that behaves well under the considered kind of reduction.

The ideas and methods used in this work derive mostly from intuitions and results presented firstly in [dC07] and [ER08]. A natural continuation of the present work could be considering the quantitative question of the execution time in λ -calculus in the sense of [dC07] from a Taylor expansion perspective. The characterization of head-normalization that we shall present has been folklore for some time. An important ispiration is [CG14], where solvability *via* Taylor expansion is considered from a call-by value perspective. For what concerns β-normalization, the result derives directly from Lemma 2.7, that has been proven firstly in [Vau17], and it is inspired also by [dCPdF11]. The result about strong normalization is new. The idea of considering non-erasing reduction derives from [dCdF16]. Our most important contribution is our approach: we give a general method to state these characterization *via* Taylor expansion that can be also extended to prove typability results for (intersection) type systems, without passing trough Girard's candidates of reducibility.

2 Results

We introduce a resource sensitive calculus following [ER08]. In this calculus the number of copies of the argument that a term uses under reduction is made explicit *via* multisets.

We define the set of *resource terms* Δ and the set of *resource monomials* $\Delta^!$ by mutual induction as follows:

$$\Delta \ni s ::= x \mid \lambda x.s \mid \langle s\rangle \bar{t} \qquad\qquad \Delta^! \ni \bar{t} ::= [] \mid [s]\cdot \bar{t}$$

If A is a set, $A^!$ denotes the set of multisets over A. Monomials are considered up to permutations and resource terms up to renaming of bound variables. Monomials are then multisets of resource terms and $\bar{s}\cdot\bar{t}$ denotes the multiset union. We write $[s_1,\dots,s_n]$ for $[s_1]\cdot\ldots\cdot[s_n]\cdot[]$. A term of the form $\langle s\rangle\bar{t}$ is called a linear application. We call *resource expressions* the elements of $\Delta^{(!)} = \Delta\cup\Delta^!$. For any resource expression e, we write $n_x(e)$ for the number of occurrences of variable x in e.

Definition 2.1. *Let $e \in \Delta^{(!)}$, $\bar{u} = [u_1,...,u_n] \in \Delta^!$ and $x \in \mathcal{V}$. We define the n-*linear substitution *of $\bar{u}$ for x in e as the following finite set of resource terms $\partial_x e\cdot\bar{u} \in \mathfrak{P}_{fin}(\Delta)$:*

$$\partial_x e\cdot\bar{u} = \begin{cases}\{e[u_{\sigma(1)}/x_1,...,u_{\sigma(n)}/x_n] \mid \sigma\in\mathfrak{S}_n\} & \text{if } n_x(e)=n\\ \emptyset & \text{otherwise}\end{cases}$$

Where $x_1,\dots,x_{n_x(e)}$ enumerate the occurrences of x in e and $\mathfrak{S}_n$ denotes the symmetry group.

The intuition is that the substitution is performed only when the number of resources in the bag is exactly the same as the number of copies of the argument called by e. For this reason the substitution is called linear.

We shall denote with $\to_\partial\subseteq \Delta\times\mathfrak{P}_{fin}(\Delta)$ the reduction relation defined contextually from the following base case: $\langle\lambda x.s\rangle\bar{t}\to_\partial \partial_x s\cdot\bar{t}$. We extend $\to_\partial$ to finite sets of terms by: $\{s\}\cup\tau\to_\partial\sigma\cup\tau$ as soon as $s\to_\partial\sigma$. Then each $\sigma\in\mathfrak{P}_{fin}(\Delta)$ reduces to exacly one (possibly empty) set of normal terms, which we write $NF(\sigma)$.

Example 2.2. *The resource version of Ω reduces to $\emptyset$:*

$$\langle\lambda x.\langle x\rangle[x]\rangle[\lambda x.\langle x\rangle[x]]\to_\partial\emptyset$$

This happens because the number of times that x is called differs from the number of arguments available.

Let M be a λ-term. We inductively define $T(M)\subseteq\Delta$, the *Taylor expansion* of M, as follows:

- if $M = x$ then $T(M)=\{x\}$;
- if $M=\lambda x.M'$ then $T(M)=\{\lambda x.s \mid s\in T(M')\}$;
- if $M = PQ$ then $T(M)=\{\langle s\rangle\bar{t} \mid s\in T(P) \text{ and } \bar{t}\in T(Q)^!\}$.

2.1 Head-normalization

The first characterization that we shall give concerns head-normalization. This result is folklore but we give a novel presentation of it following our general approach.

We denote with $H(M)$ the one-step head-reduct of M. We can extend the notion of head-reduction to resource calculus in the natural way. We shall denote as $H_\partial(s)$ the one-step head-reduct of s that is a set of resource terms.

We set $H_\partial(T(M)) = \bigcup_{s \in T(M)} H_\partial(s)$. Then we state the following commutation lemma:

Lemma 2.3. *Let M be a λ-term. Then $H_\partial(T(M)) = T(H(M))$.*

Lemma 2.4. *Let M be a λ-term. If there exists a resource term s in head-normal form such that $s \in T(M)$, then M is a head-normal form.*

Proposition 2.5. *Let M be a λ-term. If there exists $s \in T(M)$ such that $NF(s) \neq \emptyset$ then M is head-normalizable.*

Proof. We recall that $HNF(s)$ denotes the principal head-normal form of s. Since s is normalizable it is moreover head-normalizable. By definition there exist $\sigma_1, ..., \sigma_n$ such that $s = \sigma_0 \to_{\partial h} ... \to_{\partial h} \sigma_n = HNF(s)$. By Lemma 2.3, $\sigma_i \subseteq T(H^i(M))$, with $T(H^0(M)) = T(M)$, for $i \in \{0, ..., n\}$. Then by Lemma 2.4, $H^n(M)$ is a head-normal form of M (precisely the principal head-normal form of M).

□

Theorem 2.6. *Let M be a λ-term. The following statements are equivalent:*
(i) there exists a resource term $s \in T(M)$ such that $NF(s) \neq \emptyset$;
(ii) M is head-normalizable.

2.2 β-normalization

In this section we shall present a characterization of β-normalization *via* Taylor expansion. The result is essentially an application fo Lemma 2.7, that has been proved in [Vau17].

We introduce the following notion of left-parallel reduction:

$$L(\lambda x_1 \dots \lambda x_l . x Q_1 \cdots Q_k) = \lambda x_1 \dots \lambda x_l . x L(Q_1) \cdots L(Q_k)$$
$$L(\lambda x_1 \dots \lambda x_l . (\lambda x . P) Q Q_1 \cdots Q_k) = \lambda x_1 \dots \lambda x_l . (P[Q/x]) Q_1 \cdots Q_k$$

We can extend the notion of left-parallel reduction to resource calculus in the natural way. If $s \in \Delta$ we shall write $L_\partial(s)$ the one-step left-parallel reduct of s that is a set of terms.

We set $L_\partial(T(M)) = \bigcup_{s \in T(M)} L_\partial(s)$. We state now a result that extends Lemma 2.3:

Lemma 2.7. *Let M be a λ-term. Then $L_\partial(T(M)) = T(L(M))$.*

We denote with Δ^+ the set of resource terms where the empty multiset does not appear as argument of a linear application.

Lemma 2.8. *Let M be a λ-term. If there exists $s \in T(M)$ such that $s \in \Delta^+$ and s is a resource normal form then M is a β-normal form.*

Proposition 2.9. *Let M be a λ-term. If there exists $s \in T(M)$ such that $NF(s) \cap \Delta^+ \neq \emptyset$ then M is β-normalizable.*

Proof. Let $s \in T(M)$ such that there exists $u \in NF(s)$ with $u \in \Delta^+$. Then there exists $n \in \mathbb{N}$ such that $L^n_\partial = NF(s)$. By Lemma 2.8 $L^n_\partial(s) \subseteq T(L^n(M))$. Then $L^n(M)$ is the β-normal form of M. $\square$

Theorem 2.10. *Let M be a λ-term. The following statements are equivalent:*
(i) there exists $s \in T(M)$ such that there exists $NF(s) \cap \Delta^+ \neq \emptyset$;
(ii) M is β-normalizable.

2.3 Strong normalization

Inspired by [dCdF16] we want to characterize strong normalization of ordinary λ-terms *via* non-erasing reduction. The non erasing reduction is defined contextually from the following base case:

$$(\lambda x.M)N \to^{\neg e} M[N/x] \text{ if } x \in FV(M)$$

However this notion of reduction it is not enough as shown by the following example.

Example 2.11. *let $M = ((\lambda y.\lambda x.xx)z)\lambda x.xx$. Then M is by definition a non-erasing normal form, but it is not even β-normalizable*[1]*.*

Our solution is to consider a notion of non-erasing reduction that allows action at distance and prove that normalization for this kind of reduction is equivalent to standard strong normalization trough Taylor expansion. The reduction that we need is defined contextually from the following base case:

$$(\lambda x_1 ... \lambda x_n.P)Q_1....Q_n \to_\epsilon (\lambda x_1 ... \lambda x_{n-1}.P[Q_n/x_n])Q_1....Q_{n-1}$$

We denote $\to^{\neg e}_\epsilon$ the non-erasing ϵ-reduction. The only difference with $\to_\epsilon$ is that in the base case we operate the substitution only when the variable is in the body of the function. We can extend the notion of $\epsilon-$reduction to resource calculus in the natural way. We shall denote it with $\to_{\partial\epsilon}$.

[1]Interestingly, for what concerns MELL proof-nets the standard notion of non-erasing reduction is enough [dC07].

Lemma 2.12. *Let M, N be any two λ-terms. If $M \to_{\epsilon}^{\neg e} N$ then for all $t_0 \in T(N) \cap \Delta^+$ there exist $t_1, ..., t_n \in T(N)$ and a $s \in T(M)$ such that $s \to_{\partial\epsilon}^{\neg e} \{t_1, .., t_n\}$.*

Lemma 2.13. *M is normalizable through non-erasing ϵ-reduction iff there exists $s \in T(M) \cap \Delta^+$ such that $NF(s)_{\partial\epsilon}^{\neg e} \neq \emptyset$.*

Proof. ($\Rightarrow$) Since M is non-erasing ϵ-normalizable then there exists a λ-term N that is its ϵ-normal form. If we consider a reduction chain starting from M and ending in N, by Lemma 2.12 for $t_0 \in T(N)$ we can find an element $s \in T(M)$ such that $s \twoheadrightarrow_{\partial\epsilon}^{\neg e} \sigma$ with $t_0 \in \sigma$. Then there exists $s \in T(M)$ such that $NF(s)_{\partial\epsilon}^{\neg e} \neq \emptyset$. If we choose t positive, then s is a positive term, because we do not loose any information through reduction. □

Lemma 2.14. *Let $s \in \Delta$ and $\sigma, \tau \in \mathfrak{P}_{fin}(\Delta)$. If $s \to_{\partial\epsilon} \sigma$ and $\sigma \to_{\partial\epsilon}^{\neg e} \tau$ then there exists $\sigma' \in \mathfrak{P}_{fin}(\Delta)$ such that $s \to_{\partial\epsilon}^{\neg e} \sigma'$ and $\sigma' \to_{\partial\epsilon}^{+} \tau$.*

Theorem 2.15. *Let $M \in \Lambda$. The following statements are equivalent:*
(i) There exists $s \in T(M) \cap \Delta^+$ such that $NF_{\epsilon}^{\neg e}(s) \neq \emptyset$;
(ii) M is strongly normalizable.

Corollary 2.16. *M is strongly normalizable iff M is non-erasing ϵ-normalizable.*

References

[CG14] Alberto Carraro and Giulio Guerrieri. A semantical and operational account of call-by-value solvability. In *Foundations of Software Science and Computation Structures*, pages 103–118, Berlin, Heidelberg, 2014.

[dC07] Daniel de Carvalho. Semantique de la logique lineaire et temps de calcul. PhD thesis, Aix-Marseille Université, 2007.

[dCdF16] Daniel de Carvalho and Lorenzo Tortora de Falco. A semantic account of strong normalization in linear logic. *Information and Computation*, 248:104 – 129, 2016.

[dCPdF11] D. de Carvalho, M. Pagani, and L. Tortora de Falco. A semantic measure of the execution time in linear logic. *Theoretical Computer Science*, 412(20):1884 – 1902, 2011.

[ER08] Thomas Ehrhard and Laurent Regnier. Uniformity and the Taylor expansion of ordinary λ-terms. *Theoretical Computer Science*, 403(2-3), 2008.

[Vau17] Lionel Vaux. Taylor expansion, lambda-reduction and normalization. In *CSL 2017*, 2017.