Optimal Control of Oscillatory Processes in a Resistive Medium

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***Abstract* - This study addresses the optimal control problem with additional constraints for systems described by a one-dimensional oscillation equation. Theorems are presented regarding the positivity of the real eigenvalues of the spectral problem and the orthogonality of the eigenfunctions over a given interval.**

Keywords - optimal control problem, controller, solving a mixed problem, eigenvalues of the spectral problem, eigenfunction.

I. Introduction

Numerous physical and mechanical phenomena can be characterized by second-order partial differential equations. For instance, one may examine the resolution of a mixed problem characterized by the wave equation. The resolution to the boundary value problem for the wave and heat conduction equations has been presented in [1,3]. The optimal control problem of a linear system is analyzed in [2], focusing on transient process attenuation and the minimization of a quadratic functional over a finite time frame.

For systems characterized by one-dimensional wave equations, further limitations necessitate a particular methodology. This article examines the optimal control issue with further constraints for systems characterized by a one-dimensional wave equation.

II. Problem Statement

Let us consider a controlled system described by the equation:

$$\rho(x)\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x}\left(p(x)\frac{\partial y}{\partial x}\right) + u(t)\delta\big(x-\rho(x)\big) \quad (1)$$

with the initial conditions:

$$y(x,0) = \phi(x),\ y_t'(x,0) = \phi(x) \quad (2)$$

and the non-homogeneous boundary conditions:

$$\begin{cases}\alpha_{11}y(0,t) + \alpha_{12}y(l,t) = \mu_1(t),\\ \alpha_{21}y(0,t) + \alpha_{22}y(l,t) = 0,\end{cases} \quad (3)$$

Where $\rho(x)$, represents the material density of the string, $\phi(x)$ is the initial displacement of the oscillation, $\psi(x)$ is the initial amplitude of the oscillation, and $f(x)$ is a smooth function defined on $[0,l]$.

The control $u(t)$ belongs to the class of admissible controls $U = \{u(t) \le L_2(0,T), u(t) \le L\}$. The constants $\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}$ are non-zero simultaneously.

2.1 Optimal control problem

Find a control $u(t) \in V$ ch that the system (1)–(3), starting from the initial conditions (2), reaches the state:

$$y(x,T) = 0,\ \ y_t(x,T) = 0 \quad (4)$$

at the earliest time $t = T$, while satisfying:

$$d_n = \frac{[(-1)^n - 1]}{(n\pi)^2 - \ell^2} n\sqrt{\frac{\pi}{2}} cos\, 2\,\pi T. \quad (5)$$

The problem defined by (1)–(4) is referred to as the damping of oscillations. The current problem introduces an additional constraint within the oscillation damping framework.

2.2 Reduction og the problem

The solution of the mixed problem (1)–(3) for the specified control $u(t) \in V$ is sought in the form:

$$y(x,t) = X(x)T(t) \quad (6)$$

For simplicity, assume $\alpha_{12} = 0$ and $\alpha_{21} = 0$, which reduces the boundary conditions (3) to:

$$y(0,t) = \mu_1(t), \qquad y(l,t) = 0, \quad (7)$$

By substituting:

$$y(x,t) = Z(x,t) + \frac{x}{l}[-\mu_1(t)] + \mu_1(t)$$

or equivalently:

$$y(x,t) = Z(x,t) + \mu_1(t)\left(1 - \frac{x}{l}\right) \quad (8)$$

into equation (1), we obtain a new equation for $Z(x,t)$:

$$\rho(x)\frac{\partial^2 z}{\partial t^2} = \frac{\partial}{\partial x}\left[p(x)\frac{\partial z}{\partial x}\right] + f(x,t) + \rho(x)\mu_1''(t)\left[\frac{x}{l} - 1\right] \quad (9)$$

The initial conditions (2) transform to:

$$\begin{cases}Z(x,\theta) = \phi(x) + \mu_1(\theta)\left(\frac{x}{l} - 1\right)\\ Z_t'(x,0) = \psi(x) + \mu_1'(0)\left(\frac{x}{l} - 1\right)\end{cases} \quad (10)$$

and the boundary conditions (7) become:

$$z(0,t) = 0, \qquad z(l,t) = 0 \quad (11)$$

Thus, the problem (1)–(3) is reduced to solving (9)–(11).

$$\rho(x)\frac{\partial^2 z}{\partial t^2} = \frac{\partial}{\partial x}\left[p(x)\frac{\partial z}{\partial x}\right], \quad (12)$$

we seek solutions in the form:

$$z(x,t) = X(x)T(t) \quad (13)$$

Substituting (13) into (12) and separating variables gives:

$$T''(t) + \lambda T(t) = 0, \quad (14)$$

$$\frac{d}{dx}\left[k(x)\frac{dX(x)}{dx}\right] = -\lambda\rho(x)X(x) \quad (15)$$

The boundary conditions for $X(x)$are:

$$X(0) = 0, \qquad X(l) = 0.$$

III. Theorems

Theorem 1**:** If . $k(x) \ge 0$ and $\rho(x) > 0$, then the eigenvalues of the spectral problem (15)–(16) are positive real numbers.

Theorem 2**:** The eigenfunctions corresponding to distinct eigenvalues of the spectral problem (15) are orthogonal on $[0,1]$ respect to the weight $\rho(x)$.

Proof: Assume that λ_m və λ_n are eigenvalues of the spectral problem (15)-(16) such that $\lambda_m \neq \lambda_n$ Let $X_m(x)$ və

$X_k(x)$ be the eigenfunctions corresponding to these eigenvalues, such that the following identities are satisfied:

$$\frac{d}{dx}\left[k(x)\frac{dX_m(x)}{dx}\right] \equiv -\lambda_m \rho(x) X_m(x),$$
$$\frac{d}{dx}\left[k(x)\frac{dX_n(x)}{dx}\right] \equiv -\lambda_n \rho(x) X_n(x).$$

By multiplying the first equation by $X_n(x)$ and the second equation by $X_m(x)$ and thereafter subtracting the resulting equations term by term, we derive:

$$X_n(x)\frac{d}{dx}\left[k(x)\frac{dX_m(x)}{dx}\right] - X_m(x)\frac{d}{dx}\left[k(x)\frac{dX_n(x)}{dx}\right] \equiv$$
$$\equiv (\lambda_n - \lambda_m)\rho(x) X_m(x) X_n(x)$$

We shall integrate the derived equation with respect to x across the interval $[0, l]$, considering that $X_n(0) = X_n(l) = 0, X_m(0) = X_m(l) = 0$. Subsequently, we possess:

$$\int_0^l X_n(x)\frac{d}{dx}\left[k(x)\frac{dX_m(x)}{dx}\right]dx -$$
$$-\int_0^l X_m(x)\frac{d}{dx}\left[k(x)\frac{dX_n(x)}{dx}\right]dx =$$
$$= X_n(x)k(x)\frac{dX_m(x)}{dx}\Big|_0^l - \int_0^l k(x)\frac{dX_n(x)}{dx}\cdot\frac{dX_m(x)}{dx} -$$
$$-X_m(x)k(x)\frac{dX_n(x)}{dx}\Big|_0^l + \int_0^l k(x)\frac{dX_n(x)}{dx}\cdot\frac{dX_m(x)}{dx}dx =$$
$$= -\int_0^l k(x)\frac{dX_n(x)}{dx}\cdot\frac{dX_m(x)}{dx}dx +$$
$$+\int_0^l k(x)\frac{dX_n(x)}{dx}\frac{dX_m(x)}{dx} = 0$$

Thus,

$$(\lambda_m - \lambda_n)\int_0^l \rho(x)\, X_n(x) X_m(x) dx = 0$$

Since $\lambda_n \neq \lambda_m$, it follows that:

$$\int_o^l \rho(x)\, X_n(x) X_m(x) dx = 0$$

This indicates that the functions $X_m(x)$ and $X_n(x)$ are orthogonal concerning the weight function $\rho(x)$ across the interval $[0, l]$. Consequently, Theorem 2 has been demonstrated.

Given that the system of eigenfunctions $\{X_n(x)\}$ is orthogonal with respect to the weight ρ(x) on the interval $[0, l]$, it can invariably be converted into an orthonormal system. Consequently, let us presume that $\{X_k(x)\}$ constitutes an orthonormal system of eigenfunctions for the spectral problem (15)-(16), and $\{\lambda_n\}$ represents the sequence of eigenvalues.

Substituting $\lambda = \lambda_k$ into equation (14) yields the following form:

$$T_k''(t) + \lambda_k T_k(t) = 0, \qquad (17)$$

(17) which represents a homogeneous differential equation with constant coefficients. The comprehensive solution of equation (17) is

$$T_k(t) = A_k \cos\sqrt{\lambda_k}\, t + B_k \sin\sqrt{\lambda_k} t$$

where A_k and B_k are constants that remain to be ascertained.

Thus,

$$Z_k(x,t) = \left[A_k \cos\sqrt{\lambda_k}\, t + B_k \sin\sqrt{\lambda_k}\, t\right] X_k(x), \quad k = 1,2\ldots$$

functions defined here are solutions to equation (12). Since equation (12) is a linear homogeneous equation, according to a known theorem, the function

$$Z_k(x,t) = \sum_{k=1}^{\infty}\left[A_k \cos\sqrt{\lambda_k}\, t + B_k \sin\sqrt{\lambda_k}\, t\right] X_k(x),$$
$$k = 1,2.. \qquad (18)$$

is a solution to equation (12). Let us select the coefficients A_k and B_k so that the function delineated by equation (18) also fulfills the initial condition (10). Assume that the series specified by equation (18) is differentiable term by term. Then, we find:

$$\frac{\partial Z(x,t)}{\partial t} = \sum_{k=1}^{\infty}\left[-A_k\sqrt{\lambda_k}\sin\sqrt{\lambda_k}\, t + \right.$$
$$\left. + B_k\sqrt{\lambda_k}\cos\sqrt{\lambda_k}\, t\right] X_k(x)$$

If we consider the last equality and equation (12) in the context of the initial conditions (10), we can write:

$$\begin{cases} \sum_{k=1}^{\infty} A_k X_k(x) \equiv \phi(x) - \left(\frac{x}{\ell} - 1\right)\mu(0), \\ \sum_{k=1}^{\infty} B_k\sqrt{\lambda_k} X_k(x) \equiv \psi(x) - \left(\frac{x}{\ell} - 1\right)\mu_1'(0) \end{cases}$$

If we consider the last equality and equation (12) in the context of the initial conditions (10), we can write:

$$\begin{cases} \sum_{k=1}^{\infty} A_k \int_o^l \rho(x) X_n(x) X_k(x) dx = \\ = \int_0^l \rho(x)\phi(x) X_n(x) dx - \mu_1(0)\int_0^l \left(\frac{x}{\ell} - 1\right)\rho(x) X_k(x) dx, \\ \sum_{k=1}^{\infty} B_k\sqrt{\lambda_k}\int_0^l \rho(x) X_n(x) X_k(x) dx = \\ \quad = \int_0^l \rho(x)\psi(x) X_n(x) dx - \mu_1'(0)\int_0^l \frac{x}{l}\rho(x) X_n(x) dx \end{cases} \qquad (19)$$

Since the eigenfunctions $\{X_k(x)\}$ are orthonormal with respect to the weight $\rho(x)$ on the interval $[0, l]$, the following condition holds:

$$\int_0^l \rho(x) X_n(x)\, X_k(x) dx = \begin{cases} 1, & n = k, \\ 0, & n \neq k. \end{cases}$$
$$\int_0^l \rho(x)\,\phi(x) X_n(x) dx = \phi_n,$$
$$\int_0^l \rho(x)\,\psi(x) X_n(x) dx = \psi_n,$$
$$\int_0^l \left(\frac{x}{l} - 1\right)\rho(x) X_n(x) dx = \alpha_n,$$
$$\int_0^l \frac{x}{l}\rho(x) X_n(x) dx = \beta_n, n = 1,2\ldots$$

Considering that these expressions represent the Fourier coefficients of the functions $\phi(x), \psi(x), \frac{x}{l} - 1$ and $\frac{x}{l}$ with respect to the system $\{X_n(x)\}$ under the weight $\rho(x)$ over the interval $[0, l]$, we can deduce the following from system (14):

$$\begin{cases} A_n = \phi_n - \alpha_n\mu_1(0), \\ \sqrt{\lambda_n} B_n = \psi_n - \alpha_n\mu_1(0) \end{cases}$$

or

$$\begin{cases} A_n = \phi_n - \alpha_n\mu_1(0), \\ B_n = \frac{\psi_n}{\sqrt{\lambda_n}} - \frac{\alpha_n}{\sqrt{\lambda_n}}\mu_1'(0) \end{cases}$$

Substituting the computed values of A_n and B_n into equation (18) yields:

$$z^*(x,t) = \sum_{n=1}^{\infty}\left[\left(\phi_n - \alpha_n\mu_1(0)\right)\cos\sqrt{\lambda_n}\, t + \right.$$
$$\left. + \frac{1}{\sqrt{\lambda_n}}\left(\psi_n - \alpha_n\mu_1'(0)\right)\sin\sqrt{\lambda_n} t\right] X_n(x) \qquad (20)$$

Consequently, the function delineated by formula (20) constitutes the solution to problems (12), (11), and (10). We will now determine the solution to equation (9) (the inhomogeneous equation) that adheres to the homogeneous boundary and beginning conditions.

It means,

$$\rho(x)\frac{\partial^2 z(x,t)}{\partial t^2}=\frac{\partial}{\partial x}\left[k(x)\frac{\partial Z(x,t)}{\partial x}\right]+F(x,t) \qquad (21)$$

$$\left(F(x,t)=f(x,t)-\rho(x)\left(\frac{x}{l}-1\right)\mu_1''\right)$$

and the conditions

$$\begin{cases} z(0,t)=0, z(l,t)=0, \\ z(x,0)=0, z_t'(x,0)=0 \end{cases} \qquad (22)$$

Let us search for the solution of the problem (21), (22) in the form

$$z(x,t)=\sum_{n=1}^{\infty} z_n(t)X_n(x), \qquad (23)$$

where $\{X_n(x)\}$ is the orthonormal system of eigenfunctions of the spectral problem (15), (16).

If we consider,

$$\frac{\partial^2 z(x,t)}{\partial t^2}=\sum_{n=1}^{\infty} z_n''(t)X_n(x)$$

$$\frac{\partial^2 z(x,t)}{\partial x}=\sum_{n=1}^{\infty} z_n(t)\frac{dX_n(x)}{dx}$$

Then, from equation (21) we obtain:

$$\sum_{n=1}^{\infty} z_n''(t)\rho(x)X_n(x)=$$

$$=\sum_{n=1}^{\infty}\frac{d}{dx}\left[k(x)\frac{dX_n(x)}{dx}\right]z_n(t)+F(x,t)$$

or equivalently,

$$\sum_{n=1}^{\infty} z_n''(t)\,\rho(x)X_n(x)=-\sum_{n=1}^{\infty}\lambda_n\rho(x)\,X_n(x)z_n(t)+F(x,t)$$

Now, multiplying both sides of the resulting equation by $X_k(x)$ and integrating over the interval $[0,l]$ with respect to x, we get:

$$\sum_{n=1}^{\infty} z_n''(t)\int_0^l \rho(x)X_n(x)X_k(x)\,dx=$$

$$=-\sum_{n=1}^{\infty}\lambda_n\int_0^l \rho(x)X_n(x)X_k(x)\,dx\cdot z_n(t)+$$

$$+\int_0^l f(x,t)X_n(x)dx-\int_0^l\left(\frac{x}{l}-1\right)\rho(x)X_n(x)dx\mu_1''(t)$$

Considering the following integrals and properties:

$$\int_0^l \rho(x)X_n(x)X_k(x)dx=\begin{cases}1, & n=k\\ 0, & n\neq k\end{cases}$$

$$\int_0^l f(x,t)X_n(x)dx=f_n(t)$$

$$\int_0^l\left(\frac{x}{l}-1\right)\rho(x)X_n(x)dx=\alpha n,$$

$$\int_0^l \frac{x}{l}\rho(x)X_n(x)dx=\beta n, n=1,2,3,\ldots$$

From the above relations, we obtain the following equation for determining $z_n(t)$

$$z_n''(t)+\lambda_n z_n(t)=f_n(t)-\alpha_n\mu_1''(t) \qquad (23)$$

From the second condition of (22), we have the initial conditions:

$$z_n(0)=0, z_n'(0)=0 \qquad (24)$$

It can be easily shown that the solution of equation (23) with initial conditions (24) is given by:

$$z_n(t)=\frac{1}{\sqrt{\lambda_n}}\int_0^t[f_n(\tau)-\alpha_n\mu_1''(\tau)]\sin\sqrt{\lambda_n}\,(t-\tau)d\tau \qquad (25)$$

To compute the integral

$$\int_0^t \mu_1''(\tau)\sin\sqrt{\lambda_k}\,(t-\tau)d\tau$$

and substitute it into equation (25), we obtain the following expression for $z_n(t)$:

$$z_n(t)=\frac{1}{\sqrt{\lambda_n}}\int_0^t f_n(\tau)\sin\sqrt{\lambda_n}\,(t-\tau)d\tau-$$

$$-\alpha_n\left[-\frac{1}{\sqrt{\lambda_n}}\mu_1'(0)\sin\sqrt{\lambda_n}\,t+\frac{\sqrt{\lambda_n}}{\sqrt{\lambda_n}}\mu_1(t)-\right.$$

$$\left.-\mu_1(0)\cos\sqrt{\lambda_n}\,t-\sqrt{\lambda_n}\int_0^t\mu_1(\tau)\sin\sqrt{\lambda_n}\,(t-\tau)d\tau\right]=$$

$$=\frac{1}{\sqrt{\lambda_n}}\int_0^t f_n(\tau)\sin\sqrt{\lambda_n}\,(t-\tau)d\tau+$$

$$+\frac{\lambda_n}{\sqrt{\lambda_n}}\alpha_n\mu_1'(0)\sin\sqrt{\lambda_n}\,t-\alpha_n\mu_1(t)-$$

$$-\lambda_n\mu_1(0)\cos\sqrt{\lambda_n}\,t+$$

$$+\sqrt{\lambda_n}\alpha_n\int_0^t\mu_1(\tau)\sin\sqrt{\lambda_n}\,(t-\tau)d\tau$$

Thus, the solution of the system (21), (23) will be:

$$\bar{z}(x,t)=\sum_{n=1}^{\infty}\left[\frac{1}{\sqrt{\lambda_n}}\int_0^t f_n(\tau)\sin\sqrt{\lambda_n}\,(t-\tau)d\tau+\right.$$

$$+\frac{\alpha_n}{\sqrt{\lambda_n}}\mu_1(0)\sin\sqrt{\lambda_n}\,t-\alpha_n\mu_1(t)+$$

$$+\alpha_n\mu_1(0)\cos\sqrt{\lambda_n}t+$$

$$\left.+\sqrt{\lambda_n}\alpha_n\int_0^t\mu_1(\tau)\sin\sqrt{\lambda_n}\,(t-\tau)d\tau\right]X_n(x) \qquad (26)$$

The solution to the system (1)-(2) is equal to the sum of the solutions to (12), (10), and (11), along with the solutions to (21), (22), namely $(z^*(x,t)$ və $\bar{z}(x,t))$. Therefore, we have:

$$y(x,t)=z^*(x,t)+\bar{z}(x,t)+\left(\frac{x}{l}-1\right)\mu_1(t).$$

Substituting the expressions for $z^*(x,t)$ from equation (20) and $\bar{z}(x,t)$ from equation (26) into the equation for $y(x,t)$, we get:

$$y(x,t)=\sum_{n=1}^{\infty}\left[(\phi_n-\alpha_n\mu_1(0))+\frac{1}{\sqrt{\lambda_n}}(\psi_n-\right.$$

$$\left.-\alpha_n\mu_1'(0))\sin\sqrt{\lambda_n}\,t\right]X_n(x)+$$

$$+\sum_{n=1}^{\infty}\left[\frac{1}{\sqrt{\lambda_n}}\int_0^t f_n(\tau)\sin\sqrt{\lambda_n}\,(t-\tau)d\tau+\right.$$

$$+\alpha_n\mu_1(0)\cos\sqrt{\lambda_n}\,t+$$

$$+\left(\frac{x}{l}-1\right)\mu_1(t)=\sum_{n=1}^{\infty}\left[\phi_n\cos\sqrt{\lambda_n}\,t+\frac{\psi_n}{\sqrt{\lambda_n}}\sin\sqrt{\lambda_n}\,t+\right.$$

$$+\frac{1}{\sqrt{\lambda_n}}\int_0^t f_n(\tau)\sin\sqrt{\lambda_n}\,(t-\tau)d\tau+$$

$$+\sqrt{\lambda_n}\int_0^t[\alpha_n\mu_1(\tau)\sin\sqrt{\lambda_n}\,(t-\tau)d\tau]X_n(x)-$$

$$-\mu_1(t)\sum_{n=1}^{\infty}\alpha_n X_n(x)+\left(\frac{x}{l}-1\right)\mu_1(t).$$

Considering the relations:

$\sum_{n=1}^{\infty}\alpha_n X_n(x)=\frac{x}{l}-1$ and $\frac{x}{l}=\sum_{n=1}^{\infty}\beta_n X_n(x)$

the solution for $y(x,t)$ becomes:

$$y(x,t)=\sum_{n=1}^{\infty}\left[\phi_n\cos\sqrt{\lambda_n}\,t+\frac{\psi_n}{\sqrt{\lambda_n}}\sin\sqrt{\lambda_n}\,t+\right.$$

$$+\sqrt{\lambda_n}\int_0^t[\alpha_n\mu_1(\tau)]\sin\sqrt{\lambda_n}\,(t-\tau)d\tau+$$

$$\left.+\frac{1}{\sqrt{\lambda_n}}\int_0^t f_n(\tau)\sin\sqrt{\lambda_n}\,(t-\tau)d\tau\right]X_n(x).$$

Given that,

$$f_n(\tau)=\int_0^l u(\tau)\,\delta\big(x-p(\tau)\big)X_n(x)dx=u(t)X_n[p(\tau)]$$

the solution for $y(x,t)$ becomes:

$$y(x,t)=\sum_{n=1}^{\infty}\left[\phi_n\cos\sqrt{\lambda_n}\,t+\frac{\psi}{\sqrt{\lambda_n}}\sin\sqrt{\lambda_n}\,t+\right.$$

$$+\sqrt{\lambda_n}\int_0^t[\alpha_n\mu_1(\tau)]\sin\sqrt{\lambda_n}\,(t-\varepsilon)d\tau+$$

$$+\frac{1}{\sqrt{\lambda_n}}\int_0^t u(\tau)\,X_n\big[p(t)\sin\sqrt{\lambda_n}\,(t-\tau)d\varepsilon\big]X_n(x) \qquad (27)$$

Consequently, for each designated $u(t)\in V$, with $0\le p(t)\le l$ serving as the controller, the resolution of the complex system delineated by equations (1) - (3) is ascertained using formula (27). The resolved solution is a generalized solution. Assume that the function $\omega(x,t)$ is of class c^∞ on the interval $[0\le x\le l]$ and meets the following criteria:

$$\omega(0,t)=\omega(l,t)=\omega_x'(0,t)=\omega_x'(l,t)=0,$$
$$\omega(x,0)=0,\ \omega_t'(x,0)=0,\ \omega_t{}'(x,t)=0,$$
$$\omega(x,T)=0$$

Subsequently, we shall execute the integrals in a piecewise manner as outlined below:

$$\int_0^T\int_0^l\omega(x,t)\rho(x)\frac{\partial^2y(x,t)}{\partial t^2}dxdt=$$

$$=\int_0^l\left[\int_0^T\omega(x,t)\,\rho(x)\frac{\partial^2y(x,t)}{\partial t^2}dt\right]dx=$$

$$=\int_0^l[\rho(x)\omega(x,t)\frac{\partial y(x,t)}{\partial t}\Big|_0^l=$$

$$=-\int_0^T\rho(x)\frac{\partial\omega(x,t)}{\partial t}\frac{\partial y(x,t)}{\partial t}dx]dt=$$

$$=-\int_0^l\int_0^T\rho(x)\frac{\partial\omega(x,t)}{\partial t}\frac{\partial y(x,t)}{\partial t}dxdt=$$

$$=-\int_0^l[\rho(x)\,y(x,t)\frac{\partial\omega(x,t)}{\partial t}\Big|_0^T-$$

$$-\int_0^T\rho(x)\,y(x,t)\frac{\partial^2\omega(x,t)}{\partial t^2}dt]dx=$$

$$=\int_0^T\int_0^l\rho(x)\,y(x,t)\frac{\partial^2\omega(x,t)}{\partial t^2}dxdt;$$

$$\int_0^T\int_0^l\omega(x,t)\frac{\partial}{\partial x}\left[k(x)\frac{\partial y(x,t)}{\partial x}\right]dxdt=$$

$$=\int_0^T\left[\omega(x,t)k(x)\frac{\partial y(x,t)}{\partial x}\right]_0^l-$$

$$-\int_0^l\frac{\partial y(t,x)}{\partial x}k(x)\frac{\partial\omega(x,t)}{\partial x}dx]dt=$$

$$=-\int_0^T\int_0^l k(x)\frac{\partial\omega(x,t)}{\partial x}\frac{\partial y(x,t)}{\partial x}dxdt=$$

$$=-\int_0^T\left[k(x)y(x,t)\frac{\partial\omega(x,t)}{\partial x}\right|_0^l-$$

$$-\int_0^l y(x,t)\frac{\partial}{\partial x}\left[k(x)\frac{\partial\omega(x,t)}{\partial x}\right]dx\Bigg]dt=$$

$$=\int_0^T\int_0^l y(x,t)\frac{\partial}{\partial x}\left[k(x)\frac{\partial\omega(x,t)}{\partial x}\right]dxdt$$

$$\int_0^T\int_0^l\omega(x,t)\,u(t)\delta\big(x-p(t)\big)dt=\int_0^t u(t)\,\omega(p(t)t)dt.$$

For any function $\omega(x,t)\in C^\infty(o\le x\le l;0\le t\le T)$ that satisfies the conditions:

$$\omega(0,t)=\omega(l,t)=\omega_x'(0,t)=\omega_x'(l,t)=0,$$
$$\omega(x,0)=\omega(x,T)=\omega_t'(x,0)=\omega_t'(x,T)=0$$

The function $y(x,t)\in L_2[0\le x\le l];0\le t\le T$ is called generelized solution of the mixed problem (1)- (3) if it satisfies the integral equality:

$$\int_0^T\int_0^l\rho(x)\,y(x,t)\frac{\partial^2\omega(x,t)}{\partial t^2}dxdt=$$

$$=\int_0^T\int_0^l y(x,t)\frac{\partial}{\partial x}\left[k(x)\frac{\partial\omega(x,t)}{\partial x}\right]dxdt+$$

$$+\int_0^T\omega[p(t),t]\,u(t)dt.$$

IV. Conclusion

The function $z^*(x,t)$ has been defined as the solution to the system (10) - (12) by the formula (20). The solution to equation (9), the non-homogeneous equation, with homogeneous boundary and beginning conditions, has been established for $z^*(x,t)$. Consequently, for any designated $u(t)\in V$, $0\le p(t)\le l$ serving as the controller, the generalized solution of the mixed problem (1) - (3) is delineated as:

$$y(x,t)\in L_2$$
$$[0\le x\le l];0\le t\le T.$$

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