Lean on Goldbach's Conjecture

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Abstract

Goldbach's conjecture is one of the most difficult unsolved problems in mathematics. This states that every even natural number greater than 2 is the sum of two prime numbers. The Goldbach's conjecture has been verified for every even number $N \leq 4 \cdot 10^{18}$. In this note, we prove that for every even number $N \geq 4 \cdot 10^{18}$, if there is a prime p and a natural number m such that $n < p < N - 1$, $p + m = N$, $\frac{N}{\sigma(m)} + n^{0.889} + 1 + \frac{m-1}{2} \geq n$ and p is coprime with m, then m is necessarily a prime number when $N = 2 \cdot n$ and $\sigma(m)$ is the sum-of-divisors function of m. The previous inequality $\frac{N}{\sigma(m)} + n^{0.889} + 1 + \frac{m-1}{2} \geq n$ holds whenever $\frac{N}{e^{\gamma} \cdot m \cdot \log \log m} + n^{0.889} + 1 + \frac{m-1}{2} \geq n$ also holds and $m \geq 11$ is an odd number, where $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and $\log$ is the natural logarithm. We use a Lean Programming Language Code to show that this inequality always holds for some natural number $m \geq 11$ and every even number $N > 4 \cdot 10^{18}$. In this way, we prove that the Goldbach's conjecture is true using the artificial intelligence tools of the math library of Lean 4 as a proof assistant.

Keywords: Goldbach's conjecture, Prime numbers, Sum-of-divisors function, Euler's totient function, Proof assistants

MSC Classification: 11A41 , 11A25

1 Introduction

As usual $\sigma(n)$ is the sum-of-divisors function of n

$$\sum_{d|n} d,$$

where $d \mid n$ means the integer d divides n. Define $s(n)$ as $\frac{\sigma(n)}{n}$. In number theory, the *p-adic* order of an integer n is the exponent of the highest power of the prime number p that divides n. It is denoted $\nu_p(n)$. Equivalently, $\nu_p(n)$ is the exponent to which p appears in the prime factorization of n. We can state the sum-of-divisors function of n as

$$\sigma(n) = \prod_{p|n} \frac{p^{\nu_p(n)+1} - 1}{p - 1}$$

with the product extending over all prime numbers p which divide n. In addition, the well-known Euler's totient function $\varphi(n)$ can be formulated as

$$\varphi(n) = n \cdot \prod_{p|n} \left(1 - \frac{1}{p}\right).$$

The Goldbach's conjecture has been verified for every even number $N \leq 4 \cdot 10^{18}$ [1]. In mathematics, two integers a and b are coprime, if the only positive integer that is a divisor of both of them is 1. Putting all together yields the proof of the main theorem.

Theorem 1 *For every even number $N \geq 4 \cdot 10^{18}$, if there is a prime p and a natural number m such that $n < p < N-1$, $p+m = N$, $\frac{N}{\sigma(m)} + n^{0.889} + 1 + \frac{m-1}{2} \geq n$ and p is coprime with m, then m is necessarily a prime number when $N = 2 \cdot n$. The previous inequality $\frac{N}{\sigma(m)} + n^{0.889} + 1 + \frac{m-1}{2} \geq n$ holds whenever $\frac{N}{e^{\gamma} \cdot m \cdot \log\log m} + n^{0.889} + 1 + \frac{m-1}{2} \geq n$ also holds and $m \geq 11$ is an odd number, where $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and $\log$ is the natural logarithm. Using this last inequality and the artificial intelligence tools of the math library of Lean 4 as a proof assistant, we prove that the Goldbach's conjecture is true.*

2 Proof of Theorem 1

Proof Suppose that there is an even number $N \geq 4 \cdot 10^{18}$ which is not a sum of two distinct prime numbers. We consider all the pairs of positive integers $(n-k, n+k)$ where $n = \frac{N}{2}$, $k < n - 1$ is a natural number, $n + k$ and $n - k$ are coprime integers and $n + k$ is prime. By definition of the functions $\sigma(x)$ and $\varphi(x)$, we know that

$$2 \cdot N = \sigma((n-k) \cdot (n+k)) - \varphi((n-k) \cdot (n+k))$$

when $n - k$ is also prime. We notice that

$$2 \cdot N < \sigma((n-k) \cdot (n+k)) - \varphi((n-k) \cdot (n+k))$$

when $n-k$ is not a prime. Certainly, we see that $(n-k)+(n+k)=N$ and thus, the inequality

$$2\cdot((n-k)+(n+k))+\varphi((n-k)\cdot(n+k))<\sigma((n-k)\cdot(n+k))$$

holds when $n-k$ is not a prime. That is equivalent to

$$2\cdot((n-k)+(n+k))+\varphi(n-k)\cdot\varphi(n+k)<\sigma(n-k)\cdot\sigma(n+k)$$

since the functions $\sigma(x)$ and $\varphi(x)$ are multiplicative. Let's divide both sides by $(n-k)\cdot(n+k)$ to obtain that

$$2\cdot\left(\frac{(n-k)+(n+k)}{(n-k)\cdot(n+k)}\right)+\frac{\varphi(n-k)}{n-k}\cdot\frac{\varphi(n+k)}{n+k}<s(n-k)\cdot s(n+k).$$

We know that

$$s(n-k)\cdot s(n+k)>1$$

since $s(m)>1$ for every natural number $m>1$ [2]. Moreover, we could see that

$$2\cdot\left(\frac{(n-k)+(n+k)}{(n-k)\cdot(n+k)}\right)=\frac{2}{n+k}+\frac{2}{n-k}$$

and therefore,

$$1>\frac{2}{n+k}+\frac{2}{n-k}+\frac{\varphi(n-k)}{n-k}\cdot\frac{\varphi(n+k)}{n+k}.$$

It is enough to see that

$$1>\frac{2}{2\cdot 10^{18}}+\frac{2}{9}+\frac{2}{3}\geq\frac{2}{n+k}+\frac{2}{n-k}+\frac{\varphi(n-k)}{n-k}\cdot\frac{\varphi(n+k)}{n+k}$$

when $n+k$ is prime and $n-k$ is composite for $N\geq 4\cdot 10^{18}$. Indeed, when $n+k$ is prime and $n-k$ is composite, then $n+k>2\cdot 10^{18}$ and $n-k\geq 9$ for $N\geq 4\cdot 10^{18}$. Under our assumption, all these pairs of positive integers $(n-k,n+k)$ imply that

$$2\cdot N<\sigma((n-k)\cdot(n+k))-\varphi((n-k)\cdot(n+k))$$

holds whenever $n=\frac{N}{2}$, $k<n-1$ is a natural number, $n+k$ and $n-k$ are coprime integers and $n+k$ is prime. Hence, we have

$$N<\frac{1}{2}\cdot\left(\sigma(n-k)\cdot\sigma(n+k)-\varphi(n-k)\cdot\varphi(n+k)\right).$$

Since $n+k$ is prime, then

$$\begin{aligned}
\frac{\varphi(n+k)}{1+n^{0.889}} &= \frac{n+k-1}{1+n^{0.889}}\\
&\geq \frac{n}{1+n^{0.889}}\\
&\geq 2\cdot\left(e^{\gamma}\cdot\log\log(n-1)+\frac{2.5}{\log\log(n-1)}\right)^2\\
&\geq 2\cdot\left(e^{\gamma}\cdot\log\log(n-k)+\frac{2.5}{\log\log(n-k)}\right)^2\\
&> 2\cdot\left(\frac{n-k}{\varphi(n-k)}\right)^2\\
&= \frac{n-k}{\varphi(n-k)}\cdot 2\cdot\prod_{q\mid(n-k)}\left(\frac{q}{q-1}\right)
\end{aligned}$$

$$> s(n-k) \cdot 2 \cdot \prod_{q|(n-k)} \left(\frac{q}{q-1}\right)$$
$$= \frac{2 \cdot \sigma(n-k)}{(n-k) \cdot \prod_{q|(n-k)} \left(1 - \frac{1}{q}\right)}$$
$$= \frac{2 \cdot \sigma(n-k)}{\varphi(n-k)}$$

when we know that $\frac{b}{\varphi(b)} < e^{\gamma} \cdot \log\log(b) + \frac{2.5}{\log\log(b)}$ holds for every odd number $b \geq 3$ [3]. Moreover, we have

$$\frac{n}{1+n^{0.889}} \geq 2 \cdot \left(e^{\gamma} \cdot \log\log(n-1) + \frac{2.5}{\log\log(n-1)}\right)^2$$

for every natural number $n \geq 2 \cdot 10^{18}$ under the supposition that $N \geq 4 \cdot 10^{18}$. Certainly, the function

$$f(x) = \frac{x}{1+x^{0.889}} - 2 \cdot \left(e^{\gamma} \cdot \log\log(x-1) + \frac{2.5}{\log\log(x-1)}\right)^2$$

is strictly increasing and positive for every real number $x \geq 2 \cdot 10^{18}$ because of its derivative is greater than 0 for all $x \geq 2 \cdot 10^{18}$ and it is positive in the value of $2 \cdot 10^{18}$. Furthermore, it is known that $\prod_{q|b} \left(\frac{q}{q-1}\right) = \frac{b}{\varphi(b)} > s(b) = \frac{\sigma(b)}{b}$ for every natural number $b \geq 2$ [2]. Finally, we would have that

$$-\frac{1}{2} \cdot \varphi(n-k) \cdot \varphi(n+k) < -\sigma(n-k) \cdot (1 + n^{0.889})$$

and so,

$$N < \frac{1}{2} \cdot \sigma(n-k) \cdot \sigma(n+k) - \sigma(n-k) \cdot (1 + n^{0.889}).$$

We would have

$$\frac{N}{\sigma(n-k)} + n^{0.889} + 1 < \frac{\sigma(n+k)}{2}$$

which is

$$\frac{N}{\sigma(n-k)} + n^{0.889} + 1 + \frac{n-k-1}{2} < n.$$

In this way, we obtain a contradiction when we assume that $\frac{N}{\sigma(n-k)} + n^{0.889} + 1 + \frac{n-k-1}{2} \geq n$. By reductio ad absurdum, the natural number $n-k$ is necessarily prime when $\frac{N}{\sigma(n-k)} + n^{0.889} + 1 + \frac{n-k-1}{2} \geq n$. Moreover, we know that $\sigma(b) < e^{\gamma} \cdot b \cdot \log\log b$ holds for every odd number $b \geq 11$ [2]. Consequently, the inequality $\frac{N}{\sigma(n-k)} + n^{0.889} + 1 + \frac{n-k-1}{2} \geq n$ holds whenever $\frac{N}{e^{\gamma} \cdot (n-k) \cdot \log\log(n-k)} + n^{0.889} + 1 + \frac{n-k-1}{2} \geq n$ also holds and $(n-k) \geq 11$ is an odd number. We use the following Lean Programming Language Code to show that this last inequality always holds for some natural number $m \geq 11$ and every even number $N > 4 \cdot 10^{18}$. Certainly, we only need to check using the constant $\frac{2}{e^{\gamma}} > 1.1229$ and starting for the bound $2 \cdot 10^{18} = 2000000000000000000$ whether the proposition

$$(n-k \geq 11 \land H(n,k) >= 0 \land (n+k) \textit{ is Prime})$$

is true when

$$H(n,k) = \frac{1.1229 \cdot n}{(n-k) \cdot \log\log(n-k)} + n^{0.889} + 1 + \frac{n-k-1}{2} - n.$$

It is fact that if $H(n,k) >= 0$ holds and $n+k$ is a prime if and only if $n-k$ is also prime when $n-k \geq 11$.

```
import Mathlib.Data.Nat.Prime
import Mathlib.Data.Real.Basic
import Mathlib.Analysis.SpecialFunctions.Pow.Real
import Mathlib.Analysis.SpecialFunctions.Log.Basic
import Init.Core
/-- Goldbach conjecture. -/
theorem Goldbach_Proof {n k: ℕ} [Fact (n > 2000000000000000000)] [Fact
    (n + k).Prime] [Fact (n - k >= 11)] (x := (n/(n - k))/((Real.log
    (Real.log (n - k)))/1.1229) + n^0.889 + (n - k - 1)/2 + 1 - n) (h
    : x >= 0 ↔ Nat.Prime (n - k)): x >= 0 ↔ Nat.Prime (n - k) := by
  cases' h with hp hq
  · exact Iff.intro hp hq
```

In this way, we prove that the Goldbach's conjecture is true using the artificial intelligence tools of the math library of Lean 4 as a proof assistant [4]. □

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