Coloring of Graphs Avoiding Bicolored Paths of a Fixed Length

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July 13, 2021

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Abstract. The problem of finding the minimum number of colors to color a graph properly without containing any bicolored copy of a fixed family of subgraphs has been widely studied. Most well-known examples are star coloring and acyclic coloring of graphs (Grünbaum, 1973) where bicolored copies of P_4 and cycles are not allowed, respectively. We introduce a variation of these problems and study proper coloring of graphs not containing a bicolored path of a fixed length and provide general bounds for all graphs. A P_k-coloring of an undirected graph G is a proper vertex coloring of G such that there is no bicolored copy of P_k in G, and the minimum number of colors needed for a P_k-coloring of G is called the P_k*-chromatic number* of G, denoted by $s_k(G)$. We provide bounds on $s_k(G)$ for all graphs, in particular, proving that for any graph G with maximum degree $d \geq 2$, and $k \geq 4$, $s_k(G) \leq \lceil 6\sqrt{10} d^{\frac{k-1}{k-2}} \rceil$. Moreover, we find the exact values for the P_k-chromatic number of the products of some cycles and paths for $k = 5, 6$.

Keywords: graphs, acyclic coloring, star coloring

1 Introduction

The proper coloring problem on graphs seeks to find colorings on vertices with minimum number of colors such that no two neighbors receive the same color. There have been studies introducing additional conditions to proper coloring, such as also forbidding 2-colored copies of some particular graphs. In particular, *star coloring* problem on a graph G asks to find the minimum number of colors in a proper coloring forbidding a 2-colored P_4, called the star-chromatic number $\chi_s(G)$ [10]. Similarly, *acyclic chromatic number* of a graph G, $a(G)$, is the minimum number of colors used in a proper coloring not having any 2-colored cycle, also called acyclic coloring of G [10]. Both, the star coloring and acyclic coloring problems are shown to be NP-complete in [2] and [15], respectively.

These two problems have been studied widely on many different families of graphs such as product of graphs, particularly grids and hypercubes. In this paper, we introduce a variation of these problems and study proper coloring of graphs not containing a bicolored (2-colored) path of a fixed length and provide general bounds for all graphs. The P_k-*coloring* of an undirected graph G, where $k \geq 4$, is a proper vertex coloring of G such that there is no bicolored copy of P_k in G, and the minimum number of colors needed for a P_k-coloring of G is called the P_k-*chromatic number* of G, denoted by $s_k(G)$. A special case of this coloring is the star-coloring, when $k = 4$, introduced by Grünbaum [10]. Hence, $\chi_S(G) = s_4(G)$ and all of the bounds on $s_k(G)$ in Section 2 apply to star chromatic number using $k = 4$.

If a graph does not contain a bicolored P_k, then it does not contain any bicolored cycle from the family $\mathcal{C}_k = \{C_i : i \geq k\}$. Thus, as the star coloring problem is a strengthening of the acyclic coloring problem, a P_k-coloring is also a coloring avoiding a bicolored member from $\mathcal{C}_k$. We call such a coloring, a $\mathcal{C}_k$-*coloring*, where the minimum number of colors needed for such a coloring of a graph G is called $\mathcal{C}_k$-*chromatic number* of G, denoted by $a_k(G)$. By this definition, we have $a_3(G) = a(G)$. In Section 2, we provide a lower bound for the $\mathcal{C}_k$-chromatic number of graphs as well.

Our results comprise lower bounds on these colorings and an upper bound for general graphs. Moreover, some exact results are presented. In Section 2, we provide lower bounds on $s_k(G)$ and $a_k(G)$ for any graph G. Moreover, we show that for any graph G with maximum degree $d \geq 2$, and $k \geq 4$, $s_k(G) = O(d^{\frac{k-1}{k-2}})$. Finally, in Section 3, we present exact results on the P_5-coloring and P_6-coloring for the products of some paths and cycles.

1.1 Related Work

Acyclic coloring was also introduced in 1973 by Grünbaum [10] who proved that a graph with maximum degree 3 has an acyclic coloring with 4 colors.

The following bounds obtained in [3] are the best available asymptotic bounds for the acyclic chromatic number, that are obtained using the probabilistic method.

$$\Omega\left(\frac{d^{\frac{4}{3}}}{(logd)^{\frac{1}{3}}}\right) = a(G) = O(d^{\frac{4}{3}}).$$

Recently, there have been some improvements in the constant factor of the upper bound in [6, 9, 16], by using the entropy compression method. Similar results for the star chromatic number of graphs are obtained in [8], showing $\chi_s(G) \leq \lceil 20d^{3/2} \rceil$ for any graph G with maximum degree d.

We observe that the method in [6] is also used in finding a general upper bound for P_k-coloring of graphs, when k is even. This coloring is called *star k* coloring, where a proper coloring of the vertices is obtained avoiding a bicolored P_{2k}. In [6], it is shown that every graph with maximum degree Δ has a star k coloring with at most $c_k k^{\frac{1}{k-1}} \Delta^{\frac{2k-1}{2k-2}} + \Delta$ colors, where c_k is a function of k. Our result presented in Section 2 improves this result and generalizes Fertin et al.'s result in [8] to P_k-coloring of graphs for $k \geq 4$.

The star chromatic number and acyclic chromatic number of products of graphs have been studied widely as well. In [8], various bounds on the star chromatic number of some graph families such as hypercube, grid, tori are obtained, providing exact values for 2-dimensional grids, trees, complete bipartite graphs, cycles, outerplanar graphs. More recent results on the acyclic coloring of grid and tori can be found in [1] and [11]. Similarly, the acyclic chromatic number of the grid and hypercube is studied in [7]. Moreover, [12–14] investigate the acyclic chromatic number for products of trees, products of cycles and Hamming graphs. For some graphs, finding the exact values of these chromatic numbers has been a longstanding problem, such as the hypercube.

2 General Bounds

We obtain lower bounds on $s_k(G)$ and $a_k(G)$ by using the theorem of Erdős and Gallai below.

Theorem 1. *[4] For a graph G on n vertices, if the number of edges is more than*

1. $\frac{1}{2}(k-2)n$, *then G contains P_k as a subgraph,*
2. $\frac{1}{2}(k-1)(n-1)$, *then G contains a member of $\mathcal{C}_k$ as a subgraph,*

for any P_k with $k \geq 2$, and for any $\mathcal{C}_k$ with $k \geq 3$.

As also observed in [8] for star coloring, the subgraphs induced by any two color classes in a P_k-coloring are P_k-free. Using this observation together with Theorem 1, we obtain the results in Theorems 2 and 3.

Theorem 2. *For any graph $G = (V, E)$, let $|V| = n$ and $|E| = m$. Then, $s_k(G) \geq \frac{2m}{n(k-2)} + 1$, for any $k \geq 3$.*

Theorem 3. *For any graph $G = (V, E)$, let $|V| = n$, $|E| = m$ and $\Delta = 4n(n-1) - \frac{16m}{k-1} + 1$. Then, $a_k(G) \geq \frac{1}{2}(2n + 1 - \sqrt{\Delta})$, for any $k \geq 3$.*

We obtain an upper bound on the P_k-chromatic number of any graph on n vertices and maximum degree d. Our proof relies on Lovasz Local Lemma, for which we provide some preliminary details as follows. An event A_i is *mutually independent* of a set of events $\{B_i \mid i = 1, 2..., n\}$ if for any subset $\mathcal{B}$ of events or their complements contained in $\{B_i\}$, we have $Pr[A_i \mid \mathcal{B}] = Pr[A_i]$. Let $\{A_1, A_2, ..., A_n\}$ be events in an arbitrary probability space. A graph $G = (V, E)$ on the set of vertices $V = \{1, 2, ..., n\}$ is called a *dependency graph* for the events $A_1, A_2, ..., A_n$ if for each i, $1 \leq i \leq n$, the event A_i is mutually independent of all the events $\{A_j \mid (i, j) \notin E\}$.

Theorem 4 (General Lovasz Local Lemma). *[5] Suppose that $H = (V, E)$ is a dependency graph for the events $A_1, A_2, ..., A_n$ and suppose there are real numbers $y_1, y_2, ..., y_n$ such that $0 \leq y_i \leq 1$ and*

$$Pr[A_i] \leq y_i \prod_{(i,j) \in E} (1 - y_j) \tag{1}$$

for all $1 \leq i \leq n$. *Then* $Pr[\bigwedge_{i=1}^{n} A_i] \geq \prod_{i=1}^{n}(1-y_i)$. *In particular, with positive probability no event* A_i *holds.*

We use Theorem 4 in the proof of the following upper bound.

Theorem 5. *Let G be any graph with maximum degree d. Then* $s_k(G) \leq \lceil 6\sqrt{10} d^{\frac{k-1}{k-2}} \rceil$, *for any* $k \geq 4$ *and* $d \geq 2$.

Proof. Assume that $x = \lceil a d^{\frac{k-1}{k-2}} \rceil$ and $a = 6\sqrt{10}$. Let $f : V \mapsto \{1, 2, ..., x\}$ be a random vertex coloring of G, where for each vertex $v \in V$, the color $f(v) \in \{1, 2, ..., x\}$ is chosen uniformly at random. It suffices to show that with positive probability f does not produce a bicolored P_k.

Below are the types of probabilistic events that are not allowed:

- Type I: For each pair of adjacent vertices u and v of G, let $A_{u,v}$ be the event that $f(u) = f(v)$.
- Type II: For each P_k called P, let A_P be the event that P is colored properly with two colors.

By definition of our coloring, none of these events are allowed to occur. We construct a dependency graph H, where the vertices are the events of Types I and II, and use Theorem 4 to show that with positive probability none of these events occur. For two vertices A_1 and A_2 to be adjacent in H, the subgraphs corresponding to these events should have common vertices in G. The dependency graph of the events is called H, where the vertices are the union of the events. We call a vertex of H *of Type* i if it corresponds to an event of Type i. For any vertex v in G, there are at most

- d pairs $\{u, v\}$ associated with an event of Type I, and
- $\frac{k+1}{2} d^{k-1}$ copies of P_k containing v, associated with an event of Type II.

Table 1. The $(i, j)^{th}$ entry showing an upper bound on the number of vertices of type j that are adjacent to a vertex of type i in H.

	I	II
I	$2d$	$(k+1)d^{k-1}$
II	kd	$\frac{k}{2}(k+1)d^{k-1}$

The probabilities of the events are

- $Pr(A_{u,v}) = \frac{1}{x}$ for an event of type I, and
- $Pr(A_P) = \frac{1}{x^{k-2}}$ for an event of type II.

To apply Theorem 4, we choose the values of y_i's accordingly so that (1) is satisfied:

$$y_1 = \frac{1}{3d}, \qquad y_2 = \frac{1}{2(k+1)d^{k-1}}.$$

3 Coloring of Products of Paths and Cycles

The *cartesian product* of two graphs $G = (V, E)$ and $G' = (V', E')$ is shown by $G \Box G'$ and its vertex set is $V \times V'$. For any vertices $x, y \in V$ and $x', y' \in V'$, there is an edge between (x, y) and (x', y') in $G \Box G'$ if and only if either $x = y$ and $x'y' \in E'$ or $x' = y'$ and $xy \in E$. For simplicity, we let $G(n, m)$ denote the product $P_n \Box P_m$.

Theorem 6.

$$s_5(P_3 \Box P_3) = s_5(C_3 \Box C_3) = s_5(C_3 \Box C_4) = s_5(C_4 \Box C_4) = 4.$$

To prove this theorem, we start by showing that $s_5(P_3 \Box P_3) \geq 4$. Since $C_3 \Box C_3$, $C_3 \Box C_4$ and $C_4 \Box C_4$ contain $P_3 \Box P_3$ as a subgraph, this shows that at least 4 colors are needed to color these graphs. Such a coloring can be obtained as in (2) by taking the first three or four rows/columns depending on the change in the grid dimension.

$$\begin{matrix} a & b & c \\ c & a & b \\ b & c & a \end{matrix} \qquad\qquad \begin{matrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{matrix} \tag{2}$$

Theorem 7. *$s_5(G(n, m)) = 4$ for all $n, m \geq 3$.*

Proof. Note that $4 = s_5(G(3, 3)) \leq s_5(G(n, m))$ for all $m, n \geq 3$. Since there exists some integer k for which $3k \geq n, m$ and $G(n, m)$ is a subgraph of $G(3k, 3k)$, $s_5(G(n, m)) \leq s_5(G(3k, 3k))$ for some k. Hence, we show that $s_5(G(3k, 3k)) = 4$. In Theorem 6, a P_5-coloring of $C_3 \Box C_3$ is given by the upper left corner of the coloring in (2) by using 4 colors. By repeating this coloring of $C_3 \Box C_3$ k times in $3k$ rows, we obtain a coloring of $G(3k, 3)$. Then repeating this colored $G(3k, 3)$ k times in $3k$ columns, we obtain a P_5-coloring of $G(3k, 3k)$ using 4 colors. There exists no bicolored P_5 in this coloring.

In the following, we generalize the previous cases by making use of the well-known result below.

Theorem 8 (Sylvester, [17]). *If $r, s > 1$ are relatively prime integers, then there exist $\alpha, \beta \in \mathbb{N}$ such that $t = \alpha r + \beta s$ for all $t \geq (r - 1)(s - 1)$.*

Theorem 9. *Let $p, q \geq 3$ and $p, q \neq 5$. Then $s_5(C_p \Box C_q) = 4$.*

Proof. The lower bound follows from Theorem 6. By Theorem 8, p and q can be written as a linear combination of 3 and 4 using nonnegative coefficients. By using this, we are able to tile the $p \times q$-grid of $C_p \Box C_q$ using these blocks of 3×3, 3×4, 4×3, and 4×4 grids. Recall that the coloring pattern in (2) also provides a P_5-coloring of smaller grids listed above by using the upper left portion for the required size. Therefore, using these coloring patterns on the smaller blocks of the tiling yields a P_5-coloring of $C_p \Box C_q$.

Corollary 1. *Let* $i, j \geq 3$ *and* $i, j \neq 5$*. Then,* $s_5(P_i \square C_j) = 4$*.*

Proof. Since $P_i \square P_j$ is a subgraph of $P_i \square C_j$, Theorem 7 gives the lower bound. By Theorem 9, we have equality.

The ideas used above can be generalized to P_6-coloring of graphs. We are able to show the following result by using the fact $s_6(G(4,4)) \leq s_5(G(4,4)) = 4$ and by proving that three colors are not enough for a P_6-coloring of $G(4,4)$.

Theorem 10. $s_6(G(4,4)) = 4$.

Together, with Theorem 10 and $s_6(G(n,m)) \leq s_5(G(n,m)) = 4$, we have the following.

Corollary 2. $s_6(G(n,m)) = 4$ *for all* $n, m \geq 4$*.*

Similarly, Theorem 9 and Corollary 2 imply the following result.

Corollary 3. $s_6(C_m \square C_n) = 4$ *for all* $m, n \geq 4$ *and* $m, n \neq 5$*.*

Acknowledgements

The research of the second author was supported in part by the BAGEP Award of the Science Academy.

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