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Gödel mathematics versus Hilbert mathematics. II Logicism and Hilbert mathematics, the identification of logic and set theory, and Gödel's "completeness paper" (1930)

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Abstract. The previous *Part I* of the paper (https://doi.org/10.33774/coe-2022-wlr02) discusses the option of the Gödel incompleteness statement (1931: whether "Satz VI" or "Satz X") to be an axiom due to the pair of the axiom of induction in arithmetic and the axiom of infinity in set theory after interpreting them as logical negations to each other. The present *Part II* considers the previous Gödel's paper (1930) (and more precisely, the negation of "Satz VII", or "the completeness theorem") as a necessary condition for granting the Gödel incompleteness statement to be a theorem just as the statement itself, to be an axiom. Then, the "completeness paper" can be interpreted as relevant to Hilbert mathematics, according to which mathematics and reality as well as arithmetic and set theory are rather entangled or complementary rather than mathematics to obey reality able only to create models of the latter. According to that, both papers (1930; 1931) can be seen as advocating Russell's logicism or the intensional propositional logic versus both extensional arithmetic and set theory. *Reconstructing* history of philosophy, Aristotle's logic and doctrine can be opposed to those of Plato or the pre-Socratic schools as establishing ontology or intensionality versus extensionality. Husserl's phenomenology can be analogically realized including and particularly as philosophy of mathematics. One can identify propositional logic and set theory by virtue of Gödel's completeness theorem (1930: "Satz VII") and even both and arithmetic in the sense of the "compactness theorem" (1930: "Satz X") therefore opposing the latter to the "incompleteness paper" (1931). An approach identifying homomorphically propositional logic and set theory as the same structure of Boolean algebra, and arithmetic as the "half" of it in a rigorous construction involving information and its unit of a bit. Propositional logic and set theory are correspondingly identified as the shared zero-order logic of the class of all first-order logics and the class at issue correspondingly. Then, quantum mechanics does not need any quantum logics, but only the relation of propositional logic, set theory, arithmetic, and information: rather a change of the attitude into more mathematical, philosophical, and speculative than physical, empirical and experimental. Hilbert's epsilon calculus can be situated in the same framework of the relation of propositional logic and the class of all mathematical theories. The horizon of *Part III* investigating Hilbert mathematics (i.e. according to the Pythagorean viewpoint about the world as mathematical) versus Gödel mathematics (i.e. the usual understanding of mathematics as all mathematical models of the world external to it) is outlined.

Keywords: arithmetic, Aristotle, bit of information, Boolean algebra, first-order logic, Gödel, epsilon calculus, Husserl, logicism, propositional logic, ontology, Pythagoreanism, quantum logic, Russell, set theory

I INSTEAD OF INTRODUCTION: HISTORY OF LOGICISM SINCE ARISTOTLE, OR LOGIC AS ONTOLOGY

In fact, logic, or "mathematical logic" as the same discipline is called sometimes, has been assigned to mathematics only since the second half of the 19th century, but it had been formulated as a science of correct thought similar or in the framework of philosophy since Aristotle and his opposition to Plato and his ideas seeming only to double one-to-one all things (unlike the propositions meant by logic) and in turn perhaps originating from the Pythagorean mystic and sacral "Numbers" (studied by the "profane" arithmetic out of their alleged "divinity"). So, one can trace back the ancient "wandering" of concepts between or within both contemporary philosophy (and even theology) and mathematics:

The numbers of arithmetic able to be simultaneously the Pythagorean "Numbers" passed into Platonian "ideas" being only philosophical and then, into Aristotle "logic" being spontaneously "ontology", that is: "what is" or at least the "speech of what is". The conceptual wandering at issue can be realized from a contemporary viewpoint as a permanent problem about the relation of philosophy and mathematics in their foundations: maybe partly coinciding, but nonetheless being different sciences. So, arithmetic touching philosophy (and even theology) in Pythagoreanism was absolutely removed from philosophy by the newly introduced "ideas" of Platonism, but anyway partly restored in a quite unrecognizable form by means of Aristotle's logic (however recovered and rediscovered to be mathematical and thus relative to arithmetic though only in the 19th century as "Boolean algebra"[1]).

That scheme about the relation of mathematics and philosophy in their shared ancient origin needs two doctrines more: Socrates by his "human turn" in philosophy and Euclid managed to include an empirical (at least then) science as geometry into mathematics by means of the invented by him logical method, now called "deductive-axiomatic". Their influence to philosophy and mathematics assisted to be so far apart that their inherent link to seem not to exist (and even not to have ever existed).

The effect of Socrates's "problem of human" as the main one for philosophy (rather than that of Being meant eventually by Pythagoreanism as its sacral and mystic "Numbers" or rather by other pre-Socratic schools postulating ones or others "Elements" for it) can be rediscovered in Plato's ideas already absolutely emancipated from arithmetic (and thus, from contemporary mathematics), but accessible to human mind able to resolve human problems by means of them and furthermore, to organize human life and society without any relation to arithmetic or mathematics, really and empirically accessible only to few "initiates" due to its sophistication.

Euclid's geometry, demonstrating the existence of a relevant logical way for an empirical science (literally referring to the measurement of earth according to its name) to be consistently built, moved mathematics away far, far from philosophy, but analogically establishing the human mind as its creator though obeying necessary logical rules. The warning (whether real or alleged) for no one "ignorant of geometry" to come under Plato's "roof" (whether literal or metaphorical) would be natural and consistent with Socrates's "turn" since both doctrines established human mind as the only link between philosophy and mathematics, and thus, as the supreme arbitrator about any problem touching their relation since any connection between them otherwise than through and by means of human mind did not exist.

From a contemporary viewpoint, Euclid described or created geometry as a first-order logic, by which he managed to redirect geometry from physics (or empirical science) to mathematics as

[1] A dilemma of logic, considered by Hartimo (2006) in relation to Husserl and thus being relevant to the present paper.

the realm of all first-order or higher order logics in relation to propositional logic heralded by Aristotle as foundation of thought[2] and even, that of all the world as ontology. Of course, Euclid's particular achievement, as to geometry properly, is remarkable. One may say that Einstein's general relativity interpreting the physical interaction of gravitation as a geometrical and thus mathematical doctrine follows the pathway pre-charted by Euclid.

However, his invention of a general method (called "deductive-axiomatic" nowadays) for mathematics to be built as a homogeneous class of theories interpretable to be first or higher order logics[3] is even much more essential. Then, a stingy tuple of axioms (unifying both "axioms" and "postulates" of Euclid's geometry) serves to describe exhaustively the subject of the theory at issue and then, propositional logic shared by all mathematical theories is able to generate all relevant theorems in each of them.

Cantor, only in the 19th century, suggested his set theory, which could be realized in the just sketched context as a general first-order logic referring to the class of all possible mathematical theories and thus first-order logics as a class of equivalence just meant by set theory by virtue of its fundamental concept of "set". So, set theory can be all or any mathematical theories in virtue to be *mathematical*, i.e. being first-order logics.

Following the same idea (by the by, accomplished by the group "Bourbaki" step by step and volume by volume), two fundamental problems concerning arithmetic and propositional logic appear correspondingly: what are those mathematical doctrines once the foundation of set theory is granted absolutely reasonable and provable in relation to all the rest mathematical theories? In other words, can (or how) each of both be relevantly interpreted as a first-order logic under the assumption that set theory describes thoroughly the class of equivalence of all first-order logics and thus both at issue after obviously being "degenerate cases" of first-order logic? Namely:

Propositional logic interpreted simultaneously as a first-order logic is involved ambiguously to itself, or "self-referential" coinciding as the unique general and omnipresent zero-order logic with itself as a mathematical theory built according to the rules of any first-order logic[4]. The way of "degeneracy" of arithmetic to set theory is more sophisticated and not so obvious, at least in the framework of the mathematical tradition: a proof that the axiom of induction[5] (e.g., in Peano arithmetic) is the logical negation[6] of the axiom of infinity (e.g., in ZFC set theory) is necessary and rigorously justified in the previous *Part I* (Penchev 2022 October 21):

Then, the relation of set theory to arithmetic can be realized as that of non-Euclidean geometry to Euclidean geometry distinguishable from each other by a single axiom in the same list, namely the famous "Fifth postulate of Euclid", and following their shared formal scheme, now substituted

[2] For example, Oderberg (2002) discusses intensionality as intelligibility.
[3] For example, Benzmüller, Brown, and Kohlhase (2004) discuss "higher-order semantics and extensionality".
[4] For example, Gaifman (1983) discusses "self-applications" in the context of paradoxes of infinity.
[5] The axiom of induction is available in various arithmetic systems (e.g., Maliaukienė 2000; 1997).
[6] In fact, set theory with the negation of the axiom of infinity is also possible (e.g., Baratella, Ferro 1993).

by the pair of the axiom of induction versus the axiom of infinity[7]. The transition from an axiom to its negation is that special complicated way for arithmetic to originate by degeneration from set theory (just as Euclidean geometry can be seen as originating, though being counterfactual historically, from non-Euclidean geometry).

The philosophical lesson learnt by the latter pair can be continued up to the former furthermore following Riemann's ideas advocated in his dissertation allowing for both to be unified by a relevant mathematical quantity ("space curvature" or "tensor of space curvature") so Euclidean geometry to be considered as a special and unique case among the class all possible non-Euclidean geometries: being featured by *zero* space curvature. Then and in particular, Euclidean geometry can be also interpreted to be relevant to any *local* neighborhood about any point in a manifold Riemannian in general or *globally*.

Translated into the "language" of the pair of arithmetic and set theory, this would mean that arithmetic is relevant only *locally* to the *global* set theory: accordingly, the degeneration from the latter to the former should be understood as the substitution of the *global* description valid to all first-order logics (once set theory refers to their class of equivalence) to its *local* counterpart meant by arithmetic.

In fact, Gödel's completeness (1930) and incompleteness (1931) papers corresponds to the former and latter problems as they are formulated above[8], however, of course, without sharing the present context, in which both can be accordingly interpreted as follows:

The "completeness paper" elucidates that propositional logic can be equally interpreted as a first-order logic and thus in the framework of set theory as an absolutely legitime mathematical theory like all the rest. On the contrary, its incompleteness counterpart answers negatively as to arithmetic since the attempt to be interpreted as a usual first-order logic if set theory is able to describe exhaustively their class implies the Gödel-like dichotomy now realized so: either arithmetic is not a first-order logic (being contradictory to set theory literally) or if it anyway is interpreted to be that, something (though quite uncertain) misses, lacks to complement it to the standard class of all first-order logics meant by set theory.

The previous *Part I* (Penchev 2022 October 21), however, reinterprets the Gödel incompleteness statement to be an axiom rather than a theorem[9] consistently deducible from the standard axioms of arithmetic, set theory and propositional logic since the proof of the latter includes a true paradox (the famous Gödel insoluble statement) in the chain ending into the dichotomy at issue. So, it is hidden in its premises themselves and more precisely, in the direct

[7] Meaning also Quine's viewpoint linking the axiom of infinity and ω-inconsistency (Quine 1953) or in relation to Carnap's interpretation of the former (Lavers 2016).

[8] Von Plato (2018) investigates the way for Gödel to realize his specific area of problems starting from a "system of natural deductions".

[9] If the impact of Gŏdel's incompleteness theorems on mathematics is huge (e.g., Macintyre 2011), the problem about the incompleteness statement, whether an axiom or a theorem, is very essential for mathematics and philosophy of mathematics.

contradiction of the axiom of induction in arithmetic[10] (implying for all natural numbers to be *finite*) and the axiom of infinity[11] in set theory (implying for the *set* of all natural numbers to be *infinite*). Then, the finiteness of arithmetic either contradicts the infinity of set theory or it is a true part and thus incomplete to infinity (postulated by set theory) therefore demonstrating that the consideration here is consistent to (and furthermore is able to underlie and explain) the dichotomy.

The pair of arithmetic and set theory likened to that of Euclidean and non-Euclidean geometries after Riemann's unification by "space curvature" can be in turn opposed to propositional logic inherently and initially distinguishable from any first-order logic and thus, from set theory (if it is granted to be their class of equivalence) by means of its *intensionality*. This means that the intensionality[12] immanent for propositional logic is opposed to the extensionality of any first-order logic[13] and thus, to set theory being already exactly determined by the tuple of axioms relevant to any mathematical theory at issue. The approach implies that set theory, referring to any sets of any elements (e.g. in Cantor's "naive" set theory), is extensional in definition[14] just as propositional logic meaning only statements eventually relating to sets is intensional.

A problem appears as to arithmetic once the opposition of extensionality[15] versus intensionality[16] is granted and represented by set theory and propositional logic accordingly: it seems to be extensional similar to set theory, but nonetheless, distinguished from it by virtue of the Gödel dichotomy therefore implying somehow or at first glance the existence of two kinds of extensionality[17]: the one is to be a *finite extensionality* featuring arithmetic unlike the alternative *infinite extensionality* of set theory (furthermore eventually able to include the former kind of finite extensionality as a particular case).

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[10] For example, Ryll-Nardzewski (1952) discusses the "role of the axiom of induction in elementary arithmetic".

[11] One of the first comparisons of the axiom of induction with the axiom of infinity is that of Keyser (1903).

[12] Intensionality is to be absolutely distinguished from intentionality (Jacquette, Sugden 1986; Kneale, Prior 1968; Cornman 1962) as well as from identity (i.e., Jacquette 2000; Tomberlin 1984).

[13] Rheinwald (1994), Nuyen (1987), Heinaman (1984), or Lycan (1974) mean a much wider problem for naturalism originating from the opposition of intensionality to causation inherent from the physical world and thus, to any first-order logic or to their class as set theory respectively, on the one hand. On the other hand, extensionality and universality (as in *Rimscha 1981*) can generate a converse trouble also in the context of Tarski's theorem or the extensionality of truth (Shapiro 2013). On the contrary, extensionality implies causal contexts (Rosenberg, Martin 1979), and it is consistent with infinity in various versions of set theory (Sato 2009).

[14] One can emphasize the "axiom of extensionality" in set theory (e.g., Gandy 1956; 1959; Ferrari, Longo 1978; Esser 2003), where it is the "first one" in their list (though only in tradition): being only a conventional postulate, it can be rejected in set theory not less consistently (e.g., Andrews 1972).

[15] For example, as the opposition of extensionality to constructiveness (Valentini 2002), or extensionality as the restriction in naive set theory (Weber 2010),

[16] As well as "paradoxes of intensionality" (Tucker, Thomason 2011).

[17] Sylvan (2003) discusses the relation of nonexistent objects and extensionality; Quine's "policy of extensionality" (Quine 1994; Thiers 1978) is not less relevant also in the context of his new foundations (Roser 1952).

The same distinction can be realized in a quite different way as well. Both immanent, the extensionality of set theory and intensionality of propositional logic[18] can be unified by the Gödel completeness (1930) paper or as two interpretations of Boolean algebra (as this follows below). Once they have been identified, the same of both can be opposed to arithmetic and its specific extensionality not being able to be absolutely identifiable with the intensionality of propositional logic[19] and as a result, the Gödel incompleteness statement (1931) should be paraphrased already explicitly in relation to intensionality of propositional logic: the dichotomy is due the different relation of the extensionality of set theory versus that of arithmetic, both to the intensionality of propositional logic. Namely: the extensionality of set theory can be identified with the intensionality of propositional logic (at least in the sense of the Gödel completeness paper) unlike that of arithmetic not being identifiable with it in any sense[20].

Then, one can approach both Gödel papers reinterpreting them as an investigation which kind of extensionality (the finite extensionality of arithmetic or the infinite extensionality of set theory) is consistent with the inherent intensionality of propositional logic once both have been realized to be first-order logics to propositional logic as the shared "zero-order" logic of mathematics at all[21].

However, the answer, though following Gödel literally, would be partly paradoxical to the standard reading of his papers: both are consistent to the intensionality of propositional logic and thus they are reasonably interpreted to be first-order logics, and the only distinction between them opposing the finiteness of arithmetic to the infinity of set theory is irrelevant to extensionality, though it being granted admits only the infinite one, that of set theory, after the identification with it, for example in the sense of Gödel's completeness paper (1930).

That paradoxical reading can be quite discernibly illustrated if one borrows Husserl's epoché from his phenomenology and dare paraphrase it from an attitude to reality (as it is originally formulated) into a newly invented attitude to infinity[22] therefore refraining from, or withholding any judgment (mathematically, statement or conclusion) whether whatever mathematical entity is featured by finite or infinite extensionality. For example, if that entity is any set after set theory, the question whether it is a finite set or an infinite set is to be declared to be meaningless therefore erasing the distinction between arithmetic and set theory (respectively the idempotent opposition of the axiom of induction and that of infinity accordingly).

The approach can be also visualized by the pair of Euclidean and non-Euclidean geometries distinguishable only by the Fifth postulate, which now is to be "excluded" in the following sense. One considers the class of equivalence of those two kinds of geometries able to involve only the theorems in both, not needing either version of that axiom. However, the illustration can be very

[18] In the framework of the "intensionality of mathematics" as in Fefereman (1985).
[19] One may see arithmetic logically as a kind of general inscriptionalism by enumerating after comparing exceptionalism and intensionality (e.g., as Parsons 2013).
[20] For example, Auerbach (1985) investigates intensionality in the context of the Gödel theorems.
[21] For example, Malinowski (2004) deduces the concept of "inferential intensionality".
[22] Infinity is a shared fundamental concept of both mathematics and philosophy: for example, Robinson and Harré (1964) trace it in the philosophical tradition.

instructive if one reckons that (for any Riemannian manifold) Euclidean geometry is to be related to its local aspect, and non-Euclidean one correspondingly, to its global aspect.

Then, the visualization implicitly paraphrases Husserl's original "epoché" in one more way: the class of equivalence of both kinds of geometries withholds or refrains from any distinction between locality and globality: or in other words, it is "*glocal*" in definition. Immediately, two physical allusions are unavoidable as well as their unification in virtue of the shared "glocality", otherwise seeming to be absolutely separate from each other:

(1) The Standard model identifies the local and global physical space relevant to quantum mechanics so that both to be the same separable complex Hilbert space after distinguishing them initially: this is a fact since the meant separable complex Hilbert space is "flat" just as Euclidean space being also "flat" is able to be the same locally (the neighborhood about each point of it) and globally.

On the contrary, gravitational interaction after Einstein's general relativity is to be interpreted as a local "deformation" of the global "flat" Euclidean space[23] directed to recover it in the initial, "normal" state of zero space curvature for "glocality" identifying local and global spaces needs it to be "flat". So, one can visibly observe why and how much the Standard model (as far as it needs "flatness") is inconsistent with any theory of quantum gravity if it has to be in turn consistent with general relativity (needing "curvature"). In other words, the quantity of quantum gravitation should be related to the *mismatch* of the local Hilbert space to the global one therefore necessarily transcending the framework of the Standard model postulating their coincidence.

(2) Analogically, the phenomena of entanglement can be interpreted to be "glocal" inherently and necessarily. This can be illustrated by Einstein's famous metaphor of "spooky action at a distance" referring to entanglement. Indeed, it makes sense only if local space is opposed to global space, which is the usual prejudice of common sense penetrating classical physics and passed in special and general relativity after the postulate of nor exceeding the speed of light in a vacuum therefore prohibiting the simultaneous consideration of global space once local space has been granted in advance. Namely then, entanglement implies the picture of "spooky action at a distance" particularly violating that postulate and thus ostensibly seeming to be physically inadmissible just as a "ghost" in physics, acting "spookily" at any distance.

However, if one removes the hidden premises dominating implicitly due to common sense that local and global spaces are inconsistent to each other (realizing therefore otherwise the postulate of no exceeding the speed of light in a vacuum), any "spirits" can be naturally expelled from physics. For example and following the complementarity of quantum mechanics (furthermore

[23] The discussion can be linked to that about the global curvature of the universe (e.g., Yu, Wang 2016; Capozziello, De Laurentis 2015; Vardanyan, Trotta, Silk 2009; Buchert, Carfora 2008; Ichikawa, Takahashi 2006; Wang, Gong, Su 2005; Masi et al 2002; Schmidt et al 1998; Sandage 1990). Indeed, if any curvature is to be defined as a relation of parts of a whole (being furthermore linkable to Husserl's investigation e.g. as in: *Sokolowski, 1968*), this might imply for the global curvature of the universe being a single one in definition (i.e. beyond any relation valid only to true parts of it) to be zero in virtue of the fact that the global curvature would not be a relation at all and thus equivalent to zero identically.

borrowed by the Standard model about the way local and global spaces to be able to be identified), they can be analogically granted to be complementary.

Then, the boundary of light speed can be naturally reinterpreted to be relevant in fact and properly to the case of the orthogonality of local and global spaces, and involving (in general) gravitational interaction for describing the case of their eventual non-orthogonality however only in terms within local space limited by the boundary of light speed in definition:

The following visualizing metaphor can be useful: if one attempts to stick surfaces or volumes in a place less than them being "unfolded" (that is: "flat"), they would be "folded" or "curved" just as Riemannian space is "folded" or "curved" to Euclidean space or respectively pseudo-Riemannian space to Minkowski space. Einstein's general relativity involving just pseudo-Riemannian space to explain the nature of gravitation as being due to the "curvature" of Euclidean space assists for that metaphor of "stuffing" (i.e. meaning the "stuffing" of a greater subspace of Euclidean space into a less one because of the postulate of not exceeding the speed of light in a vacuum).

According to the above illustration, one can state that all global space (figuratively as if out of the boundaries of local space outlined by the limit of light speed) is available within its "dual" local counterpart as its "folding", "curvature" or "force of gravitation". Thus, the real domain of Minkowski space is mapped into its imaginary area just as pseudo-Riemannian space[24], or speaking more or less loosely, that the exceeding of light speed is physically represented in the local space of our empirical or experimental experience under the unrecognizable form of gravitation[25].

(3) Finally, the separated considerations of pseudo-Riemannian space (1) and entanglement (2) can be unified just in virtue of the shared and elucidated above structure to reconcile local and global spaces even in the case where the latter is "curved" rather than only in the trivial case for both to be "flat" though the nature of local and global spaces seems to be quite different in (1) in comparison to (2)

Indeed, the former case means a *finite*-dimensional (whether Euclidean or Minkowski space, whether Riemannian or pseudo-Riemannian, as local and global spaces correspondingly), and the latter case refers to an *infinite*-dimensional space (such as that Hilbert space of quantum mechanics relating local and global spaces by entanglement).

Furthermore, one can immediately notice that Fourier transform (respectively, its reverse counterpart) is able to connect both cases and then physically interpretable as the substitution of "time" (as to the case of a finite-dimensional space, or "finiteness" at all) by "frequency" (as to

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[24] The transformations between Minkowski space, pseudo-Riemannian space, the separable complex Hilbert space of quantum mechanics or the qubit Hilbert space of quantum information are discussed in detail in another paper (Penchev 2022 February 4),

[25] One can discuss locality versus globality as a single one versus two dual directions, and then: general relativity as the representation of globality into locality. That is: the real domain of Minkowski space associable with speeds exceeding that of light in a vacuum is to be mapped onto the imaginary one associable with speeds less than that of light in a vacuum and that mapping is pseudo-Riemannian space meant by general relativity. The idea is suggested for the first time in another paper (Penchev 2013).

the case of an infinite dimensional space, or "infinity" at all) therefore creating a general model for unifying finiteness and infinity applicable to the pair of arithmetic and set theory from the (shared by both) viewpoint of propositional logic[26].

The physical interpretation of Fourier transform by the pair of time and frequency can be complemented in the area of quantum mechanics as well as in ontology or in the history of philosophy. Indeed, quantum mechanics unifies the *discrete* description of quantum entities and systems "by themselves", forced to be discrete by the fundamental Planck constant, with their readings measured by the macroscopic apparatus obeying the *smooth* differential equations of classical physics:

The discrete description corresponds to the member of "frequency" in the above pair of quantities, and the smooth description, to its counterpart of "time". The unification of both descriptions, inherent for quantum mechanics, but opposing it to classical mechanics, can be in turn realized by one more "epoché" similar to Husserl's one ("to reality" in original, and once paraphrased above to be "to infinity"): now "to time" and implying derivatively an independence of Fourier transform, or respectively the unification of the discrete and continuous (smooth): a unification definitive for quantum mechanics.

As to ontology and history of philosophy, a kind of "destruction" or "deconstruction" (or "reconstruction", in fact) in relation to Aristotle's logic and tracing it back as "ontology"[27] to its origin: by revising Pythagoreanism or Platonism. Indeed, Plato himself opposes his "ideas" being out of time to the temporal "things" corresponding to them and changing themselves in the course of time.

On the contrary, logic, moreover interpreted to be ontology, erases the opposition of "ideas" and "things": it introduces that "epoché" to reality (as this will be articulated by Husserl much later) by its ability to identify things and ideas (in fact, also available in advance in any natural language).

The "destruction", 'deconstruction", or "reconstruction" can be continued still back, up to the origin of philosophy by Pythagoreanism: and its concept of both empirical numbers and sacral Numbers can be revealed in the Platonist doubling of things and ideas again identified by Aristototelian logic. Then, the contemporary opposition and unification of logic, set theory, and arithmetic in the foundations of mathematics can serve for the reconstruction at issue, on the one hand, but on the other hand, as the ground and pathway to Hilbert arithmetic as the basis of Hilbert mathematics (intended to be the proper subject of the next *Part III* of the paper).

In other words, two aspects of the syncretic Numbers of Pythagoreanism can be seen separately realized and further developed in the doctrines of Plato and Aristotle: accordingly, the doubling of the same, and then, the identification over again of the same after its doubling by Plato. One might think of that reconstruction ostensibly doubling and identifying the same as a bad example of

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[26] Väänänen and Wang (2015) suggest an alternative approach.

[27] Furthermore linking "ontological commitment" and intensionality (e.g., Jubien 1972) or arithmetic (Hodes 1984) in a way quite relevant to the context of the present paper; dually, "intensionality and epistemic justification" (Bondy 2013).

metaphysical speculation, a meaningless wordplay absolutely rejected by and after Wittgenstein (etc.). Not at all and here is why (properly, only in the present context):

The concept and quantity of information and especially its unit of a bit (as well as its generalization in quantum mechanics, correspondingly "quantum information" and "qubit") can be interpreted as an analogical pair of successive identification and doubling. Indeed, "bit" means two complementary oppositions (though common sense's prejudice suggests for it to be a single one), which can be naturally realized as: (1) identification versus doubling; (2) both alternatives after doubling.

Philosophy questions about the being so that it is able to be the basis of itself by itself in a consistent way (i.e., without generating any contradiction after resolving that problem). In fact, Aristotle's innovation of propositional logic as ontology and relevantly borrowed from both theology and science for millennia after him is a successful answer to that fundamental philosophical puzzle. Husserl's phenomenology grounded on an "epoché to reality", Russell's logicism[28] as to the foundations of mathematics[29], both papers of Gödel (1930; 1931)[30] reinvented more or less Aristotle's ancient solution.

Nonetheless, it can be deepened and for example illustrated as above by a reconstruction of its origin from the doctrines of Pythagoras and Plato. Furthermore, that new approach to the way of philosophy, by which it is able to be *causa sui*, can be grounded on the quite contemporary concept of information and its unit of a bit. Anyway, if one attempts to reconcile both considerations though historically gapped by more than two millennia, a bit of information can be identified as that fundamental philosophical problem together with its solution. It consists in the understanding of a bit of information "ontologically": this means, as a "unit of *question*": indeed, an elementary "yes - no" question. One may say rather loosely that the being by itself and *causa sui* is its own question, or questionability itself: speaking in a "bad speculative manner" borrowed from German classical philosophy, the "being answers the question of itself by what it is by itself".

Then and further, one can continue by suggesting an initial, fundamental and philosophical bit of information, eventually originating from scientific transcendentalism and revealable in Husssel's phenomenology once it is realizable as a form of transcendentalism, or for example, in Husserl's "epoché" (also comparable with Hegel's synthesis[31]), paraphrased in the present paper in many ways. Indeed, phenomenological attitude opposed to natural one and proclaimed by Husserl can be thought in a Hegelian way to synthesize both "subject" and "object", "idea" and "thing", "mind" and "the world" corresponding to the Cartesian "mind" and "body". Then, if one wishes to interpret "epoché" in a natural attitude (being inherent for science and passing from it to

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[28] Russell's logicism is a very widely discussed subject, and closer to the present context (e.g.) by Gandon (2012; 2008; 2008a), Ferreirós (2009), Gilmore (2005), Clark (1993), Byrd (1989), Grattan-Guinness (1984; 1974).

[29] For example, Walsh (2014) relates logicism to arithmetic as Hochberg (1956) up Peano, or McLarty (2011), to Emmy Noether's contribution.

[30] Discussed (e.g.) by Weiermann (1998).

[31] For example, as in Lampert (1988).

"scientific transcendentalism"), it would be transformed into the "philosophical bit of information" at issue (Penchev 2021 August 24).

Then, the same "philosophical bit of information" can be discovered in the foundations of quantum mechanics, in Bohr's complementarity, in wave - particle duality, in the choice of Hilbert space (as the basic mathematical formalism of quantum mechanics) and its inherent duality, etc. What is shared in all cases as supposedly originating from the philosophical bit of information is that elementary mathematical structure which a bit is able to represent.

Once it has been extracted as an essence and lesson from them, Peano arithmetic can be analogically doubled in virtue of the same elementary mathematical structure to resolve Gödel's original dichotomy about the relation of arithmetic to set theory: either incompleteness or contradiction.[32] Indeed, any bit of information is either incomplete (to both alternatives) or contradictory (to the other alternative).

The uncertainty embedded in the dichotomy is the ambiguity of set theory. It can be interpreted as a doctrine of actual infinity and then, it contradicts arithmetic because of the axiom of induction implying for all natural numbers to be finite. However, set theory can be not worse realized as a general doctrine about all sets whether finite or infinite. If the latter is the case, arithmetic is only an incomplete set theory as a true part of it, referring to finite sets alone.

II HUSSERL'S PHENOMENOLOGY AS LOGICISM AND NEO-PYTHAGOREANISM

The reading of Husserl's phenomenology admits different interpretations, for example, by identifying it as a philosophy of mathematics[33] able to be "first philosophy"[34] and thus belonging to the classes of doctrines usually enumerated to be Pythagoreanism. The pathway of Husserl to the doctrine of phenomenology had started from his "Philosophy of arithmetic" (Husserl 1891) attempting an approach to arithmetic "in natural attitude"[35] if one reinterprets it by his later ideas and therefore opposed to them (as Husserl himself did[36]). That radical change or exchange of his viewpoint to the foundation of arithmetic (being the natural basis of mathematics) corresponded chronologically and essentially to Whitehead and Russell's "Principia mathematica" or to Gödel's

[32] One can speak of "quantum incompleteness" (e.g. as Torza 2020; Gonzalez-Mestres, Bravina, Foka, Kabana 2017; Held 2012; Tucker, Thomason 2011; Hall 2004; Garola 1994; 1992; Barut, Božić, Marić 1988; Redhead 1987; Koç 1980; Suppes, Zanotti 1974), or respectively "quantum completeness" (e.g. as Jurić 2018; Held 2015; 2012; Hofmann, Marc 2015; Lunin 2008; Garola, Sozzo 2004; Pulmanova 1996; Prugovečki 1991; Rastall 1981; Jones 1977; Fine 1974), or comparing them (Božić, Marić 1998; Andås, Gjøtterud 1993) in a more generalized sense after that dichotomy if the "contradiction" in it is interpreted as quantum complementarity as well as comparing extensionality with underdetermination or indeterminacy. (Solomon 1990). Penrose (2011) relates the philosophical essence of Gödel's approach to the laws of physics at all.

[33] For example, Haddock (Hartimo 2012; 2006; 1997; 1987).

[34] For example, McCarthy (1972) discusses Husserl's phenomenology as both philosophy of mathematics and ontology.

[35] For example, Hartimo, Okada (2016).

[36] For example, in Tieszen (1994) Tragesser (1984) or in Gödel's way to Husserl (Livadas 2019; Atten 2015; 2001; Tieszen 2012; Cassou-Noguès 2007; Hauser 2006; 1998a; 1992; Liu 2010), incl. in his unpublished papers on foundations of mathematics (Tatt 2001).

papers elucidating both mathematical and meta-mathematical relations of set theory sequentially to propositional logic and arithmetic being the other two theories fundamental for mathematics.

Sharing the same subject, one can utilize Husserl's phenomenology, before that, being reinterpreted according to his personal pathway to it, as a doctrine within philosophy of mathematics[37], to be the key for the decipherment, the "Rosetta stone" for Gödel's papers: an intention rather extraordinary or even ridiculous at first glance since they are quite not esoteric; on the contrary, they claim to be supported by detailed proofs not admitting any hidden meaning or sense in definition.

However, if one grants the main result of the *Part I* of the paper (Penchev 2022 October 21) for the Gödel incompleteness statement in his latter paper ("Satz VI") to be an axiom only disguised as a theorem, this implies the necessity of its decipherment regardless of the seeming lack of any deeper meaning hidden behind the ostensibly quite clear one. The more general idea about the unraveling of deeper and quite different meanings penetrating seemingly absolutely comprehensible texts can be found in the philosophical doctrine of Marx, Freud's psychoanalysis, hermeneutics or its philosophical extrapolations by Heidegger and Gadamer belonging to the same epoch as his studies.

Of course, the investigation of mathematical theorems as cultural or psychological facts seems to be a rather extraordinary enterprise if one has not revealed and proved in advance that they should be realized rather or even first of all as the latter rather than as the former: now, the previous *Part I* of the paper calls to be understood justifying the utilization of Husserl's phenomenology as a neo-Pythagorean kind of philosophy of mathematics sharing the ancient Aristotelian innovation of logic as ontology for deciphering Gödel's papers.

Nonetheless, nothing besides an only analogical "unraveling attitude" to ostensibly clear texts will be borrowed from Marx's or Freud's doctrines and even from Gadamer's hermeneutics. Anyway, a similarity to Heidegger's criticism in relation to European philosophy and culture can be shared, furthermore quite relevant in the context of Husserl's phenomenology as its origin.

In a sense, Husserl's conceptual twist[38] in "Logical investigations" after "Philosophy of arithmetic" can be partly interpreted as a repetition of Aristotle's logical revision of Platonism directed to ontology after doubling the empirical world of temporal "things" by eternal "ideas".

Anyway, "Philosophy of arithmetic" once the opposition of things and ideas has been granted tries to reveal the origin of arithmetic among the empirical world of temporal things rather than following any form of Platonism in mathematics. Also later than the years of "Logical investigation", Husserl's phenomenology would be featured by a special relation to time, quite different or even opposite to that of Platonism and approaching Bergson's temporal reading of transcendentalism and his fundamental concept of "durée" seeming as an oxymoron in classical

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[37] A viewpoint though more or less implicit in Willard's paper (1980; 1984).

[38] For example, Centrone 2010 or semiotically, Byrn 2017; that "twist" corresponds to the period of Hussel's lectures in 1896 (Hartimo 2012; Rollinger 2003) and Huuserl's critique to psychologism (e.g., Hartimo 2013; Hopkins 2011; Meiland 1976); though Gotesky (1938) reveals psychologism in "Logical Investigations" as well. Anyway, Husserl's logical and philosophical viewpoint is criticized by Russel (e.g., Roy 1995).

Platonism such as the idea of time or temporality. So, Husserl accomplished the reconciliation of things and ideas in logic by means of that specific phenomenological (or purely psychological) time, which can be also schematically represented as a kind of "subjective" logic after a conventional transition from classical Platonism to the "subjective" Platonism of consciousness in advance so that phenomenological time to be quite relevant to it[39].

That special phenomenological attention to time can be found in both philosophy and psychology as "rigorous science", in the "stream of consciousness", "transcendentality of consciousness", etc., rather than in Husserl's explicit studies of time or in comparisons with Heidegger's doctrine. Then, it can be traced back or projected into "Philosophy of arithmetic" after exchanging its "explained" and "explaining": from arithmetic by psychology to psychology by arithmetic[40]. An analogue of it can be researched in Gödel's concept of time, or its inherent incompleteness especially in the context of the similar incompleteness of arithmetic to set theory.

As to its counterpart, the mutual completeness of logic and set theory according to his former paper (1930), Husserl's conception of a few reductions, eidetic, phenomenological (psychological), and transcendental, may be more instructive. Indeed, his "eidetic reduction" can be understood to reveal the unambiguous intensional "correlate" of any extensional content or collection of entities granted to be given in advance. Particularly, it is able to represent any set of whether a finite cardinal number or an infinite cardinal number as a characteristic property definitive for the set, i.e. a proposition though composite and arbitrarily complicated.

In fact, Husserl's "eidetic reduction", rather under the name of "abstraction", or "method of abstraction" penetrates all branches of mathematics, relates them to "propositional logic" and finally allows for all mathematics to be relevantly interpreted as different "mathematical theories" rigorously defined to be first-order or higher-order logics and represented in the "deductive and axiomatic method" unifying all of them: even in its name, where "deductive" can be related to propositional logic, and "axiomatic", to a specific tuple of axioms able to feature any given mathematical theory as a first-order logic. Furthermore, what is called "eidetic reduction" had allowed for Aristotle to reform Platonism by means of logic into ontology: an invention able to unify rather paradoxically theology, philosophy, mathematics, and even science in the Middle Ages.

So, the sense of Husserl's "eidetic reduction" is properly related to his concept of "epoché", to "phenomenological reduction" (also "psychological reduction"), and then, to "transcendental reduction"[41] in the final analysis, and interpreting philosophical transcendentalism: rather than in its definition because it existed and had applied in many areas a long time before him though under different names.

Just "epoché" is crucial since it allows for articulating the Aristotelian approach to logic to underlie philosophy (especially after Platonism) as ontology. Epoché pioneers the pathway for

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[39] For example, Hefferman (1989).

[40] One can also mean Frege's criticism to "Philosophy of arithmetic" (e.g. Hill 1994) or the comparison of Husserl and Frege (Føllesdal 1994; Haaparanta 1988)

[41] Or (alternatively) following the "logic of transcendental reduction" (Stapleton 1982).

both Platonist counterparts of "things" and "ideas" to be indistinguishably unified as "phenomena" therefore embedding in Kantian "phenomena" another viewpoint and deepness to them along with the original one. The "phenomena" of transcendentalism also mean both Platonist "things" and corresponding to them "ideas", but rather in relation to the subject's sensual perceptions, and the analogical term of Husserl's phenomenology relates their unity to logic after "Logical investigations" therefore repeating markedly and articulately a reconstructed way for Aristotle to achieve his fundamental discovery after Plato's doctrine.

If Husserl's epoché is paraphrased as many times above (where that transfer is justified in detail) "to infinity" (from its original sense "to reality"), it is able to elucidate Russell's logicism[42] as a literal interpretation or reinterpretation of Aristotle's ideas to philosophy and the foundations of mathematics. Then, Husserl's "phenomenon" can be also immediately reinterpreted in terms of set theory as abandoning the analogical "natural attitude" in set theory or respectively, in mathematics as a collection of first-order logics, and consisting in the explicit opposition of finiteness and infinity, for example, emphasized by the concept of *actual* infinity understood as the ultimate and unambiguous result of any *finite* process if it is granted to be continuable *ad lib*.

One can describe "mathematical phenomenon", relevant also to Husserl's original "phenomenon", after the newly introduced "epoché to infinity"[43]. It can be related to the "restored Eden" of Fermat arithmetic yet not known the "original sin of infinity", and thus yet not "expelled from Paradise". That kind of arithmetic, particularly, cannot be distinguished in any way from set theory since both concepts of finiteness and infinity opposing and contradicting each other are not available whether because they had not arisen yet as in Fermat's age or because they are already unified by virtue of that "epoché to infinity".

Being inherently arithmetical, that "mathematical phenomenon" can be identified with the Numbers of Pythagoreanism, as the foundations of the world rather than those of mathematics alone. Anyway, it has been realized quite otherwise, by means of logic and ontology since Aristotle's age, abandoning absolutely the initial extensionality of arithmetic and the numbers "pregnant" with the "original sin of infinity" though explicitly "consummated" only by Cantor's set theory[44], that is: more than two millennia later.

Then, Russell's logicism is that doctrine restoring Aristotle's inceptive solution of the same problem in fact, though in relation only to mathematics nowadays[45]. Once Husserl's "epoché" has been introduced as a hermeneutical key or "Rosetta stone", the pair of Gödel's papers (1930; 1931) can be unified in an inseparable whole in favor of Russell's logicism in the thalweg of Aristotle's track. However then one needs both Gödel completeness (1930) and incompleteness (1931) to be

[42] For example, Kraal (2014), Proops (2006), Radner (1975) investigate its philosophical foundations.
[43] If one relates mathematics to theology or ontology by infinity (as e.g., Cortese 2015; Tapp 2011; Bussotti, Tapp 2009; Drozdek 1995; Le Blanc 1993), various biblical metaphors could be linked. Morgan (2011) pays attention to the "significance of the mathematics of infinity for realism". The comparison of Kant's and Husserl's concepts about experience and infinity (Tengelyi 2005), Hegel's logic and infinity (Houlgate 2005), or Husserl's pre-logical theory of experience (Lohmar 2002) would be also relevant.
[44] Following (e.g.) the viewpoint of Jahnke (2001).
[45] For example, Landini 2014.

reinterpreted to be complementary to each other rather than valid simultaneously in the following rigorous meaning:

They have to be related to each other as the two complementary oppositions of the same structure of a bit of information. Namely: the completeness paper (1930) is to be granted as referring to the choice itself of any elementary choice of a bit, but selecting the state of "no choice" or that in "Eden and before the original sin" though expressly meaning set theory, but interpreting it as a doctrine of sets at all, regardless of whether infinite or finite. On the contrary, the incompleteness paper means the state of the "eaten apple", i.e., where the choice itself has been chosen as a necessary condition of its two alternatives, which are already explicitly indicated: arithmetic (finiteness strictly without infinity) versus set theory (including infinity though eventually along with finiteness).

Then, the necessary complementarity of the two papers should be understood as follows. The alternative (i.e., the literal logical negation) of the former paper is the necessary condition of the latter paper. The "logical Eden" of the completeness paper, in which finiteness and infinity had not been opposed to each other, was abandoned after the initial choice of the choice itself. Once the "original sin has been consummated" in that way, only then, the peccable dichotomy heralded by Gödel is unavoidable: either incompleteness ("death" in a generalized biblical meaning) or contradiction.

One might think that the permanent utilization of Old Testament metaphors as to mathematical theories or theorems or in the context of Gödel is absolutely inappropriate: not at all, it is intentionally emphasized and here is why. The arguments in favor of their utilization can be divided into two groups. The one refers to the suggestion that the Gödel incompleteness statement is an axiom grounded in the general organization of cognition in Modernity. Then, the basis of that organization is to correspond to Christianity, particularly to the biblical parable of cognition as the "original sin", because of which the first people, Adam and Eve, and thus humankind have been expelled from Eden.

The other group of possible tenets for the utilization of the Old Testament metaphors refers to the fact that Gödel (as well as Einstein meaning their shared residence and friendship in Princeton) and Husserl are Jews more or less penetrated by Jewish culture and religion. Judaism, though recognizing the Old Testament as Christianity, rejects the New Testament, respectively and particularly its ideas about the Redemption of the original sin by the Savior Jesus Christ (only according to the belief of Christianity). So, the research of whether "incompleteness" (to say, after the "original sin") or "epoché" (to say, before it or in Paradise) would be natural for the attitude of the culture of Judaism, to which both Husserl and Gödel may be enumerated. Accordingly, philosophical or mathematical ideas relevant rather to the Redemption would be foreign to them.

Aristotle's original doctrine, Husserl's phenomenology, and Russell's logicism in mathematics as well as Gödel's "legal protection" advocating it[46] (if one adopts the viewpoint to both completeness and incompleteness papers as in the present *Part II* of the study), all of them can be unified and opposed to Hilbert mathematics based on Hilbert arithmetic (introduced in much more

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[46] For example, Hellman (1981) discusses the relation of Gödel's incompleteness theorems and logicism.

detail in other papers: e.g., *Penchev 2021 August 24*) if one distinguishes Pythagoreanism in a wide sense[47] (i.e. being able to include logic and ontology within its scope) from Pythagoreanism in a more narrow sense, emphasizing just arithmetic (i.e. a certain first-order logic if one uses contemporary concepts)[48].

So, the former four doctrines or theories would be in the scope of Pythagoreanism in that wide sense at issue, in which it is related to logic rather than to arithmetic as the original Pythagoreanism did. Nonetheless, Hilbert arithmetic means rather to unify both wide and narrow senses of Pythagoreanism (as they are defined above) by adding a dual and anti-isometric counterpart of Peano arithmetic especially in order to complement it as a formal structure to that of set theory or propositional logic, particularly to overcome its "incompleteness" in the sense of Gödel.

Though Hilbert arithmetic as the ground of Hilbert mathematics is intended to be the proper subject in the next *Part III* of the paper, its relation to logicism whether in Russell's (Gödel's) or Husserl's version discussed in the present section can be rather impressively illustrated by the interpretation of the Schrödsinger equation, fundamental for quantum mechanics, in the framework of Hilbert arithmetic. It means a fact seeming to be trivial, namely isometry (though in the form of anti-isometry) of both dual Peano arithmetics if they are interpreted by Hilbert arithmetic in a wide sense as corresponding derivatives of the same wave function (e.g. Penchev 2021 April 12; 2021 August 24).

Then, the wide and logical understanding of Pythagoreanism to its generalized interpretation in Hilbert mathematics by Hilbert arithmetic can be reduced (even in an exact and rigorous meaning) to the "nonstandard bijection" (e.g. Penchev 2022 June 30): $P^{+} \otimes P^{-} \leftrightarrow P^{0}$, in which can be distinguished two opposite directions and their corresponding structures corresponding to Hegel's "dialectical synthesis", on the one hand, or to a "bit" of information, on the other hand. If one interprets that "nonstandard bijection" to be anyway a bijection, this would correspond to the identification of logicism as a kind of Pythagoreanism[49] called here "Pythagoreanism in a wide sense". However, if one differs the "standard bijection" from the "nonstandard bijection" (in which the two directions are complementary to each other, in particular) thus emphasizing the mismatch of Pythagoreanism in a narrow sense from that in a wide sense, the former has to be differentiated from logicism.

The generalizing and fundamental structure of a bit of information (above meant as the "philosophical bit of information") can be realized as that relevant to philosophy of mathematics, and referable to the algebraic structure of propositional logic as Boolean algebra. Boolean algebra can be decomposed into two dual anti-isometric Peano arithmetics, each of which is a well-ordering idempotently opposed to that of its dual counterpart. In the conventionally reverse (i.e. reverse to that described in the previous sentence) direction of the nonstandard bijection, the

[47] Even following Hilbert's initial ideas about the foundations of logic and arithmetic (Hilbert 1905).

[48] Ulrich (1997), Majer (1997) as well as Da Silva (2016) suggest a comparison of the viewpoints of Husserl and Hilbert on completeness, Hartimo (2018; 2007) links Husserl's completeness to the concept of manifold, and Da Silva (2000) considers Husserl's "two notions of completeness".

[49] For example, Kolman (2015) elucidates "logicism as making arithmetic explicit".

"Hegelian synthesis" of those two dual Peano arithmetics (therefore canceling the opposite well-orderings of each of both) generates the structure of Boolean algebra traditionally interpreted to be identifiable with propositional logic.

One can immediately notice that the usual understanding of both arithmetic and set theory to be two mathematical theories[50], i.e., first-order logics to propositional logic along with any other mathematical theories though granted or only postulated to be the most fundamental ones, is substituted now to be algebraically derivative from Boolean algebra sharing the structure of a bit information:

The conclusions of both Gödel papers (1930; 1931) turn out to be trivial corollaries of the structure of a bit of information after that substitution accordingly identifying a bit of information before and after choice (standardly distinguished as the Boolean structure of propositional logic, granted to be the shared zero-order logic for all mathematical theories in definition, from the isomorphic also Boolean structure of set theory but now related to the *class* of all mathematical theories as first-order logics) and meant by the completeness paper (1930) versus either the "incompleteness" of either alternative of a bit to both alternatives together or the contradiction of each of them to the other one, following the latter paper (1931). That approach will be developed in detail in relation to both papers (1930; 1931) in the next section.

The Schrödinger equation can be interpreted thoroughly in the framework of Hilbert arithmetic, or more precisely by the equivalence (or complementarity) of it in a narrow sense and in a wide sense. Indeed, both anti-isometry Peano arithmetics in Hilbert arithmetic in a narrow sense share the same elementary step of a unit ("± 1") according to their isometry (even in the case of anti-isometry). If one substitutes the only one (either) of the two dual anti-isometry copies of Peano arithmetic by its counterpart in Hilbert arithmetic (i.e., by the qubit Hilbert space[51]), the step of one unit remains the same though now being between any two successive qubits. The discrete step of a unit corresponds to the first time derivative ("$\frac{\delta \Psi(x,y,z,t)}{\delta t}$") in the case of Hilbert arithmetic in a narrow sense (Peano arithmetic), but to the second space derivative ("$\frac{\delta^2 \Psi(x,t)}{\delta x^2}$" or "$\frac{\delta^2 \Psi(x,y,z,t)}{\delta x^2} + \frac{\delta^2 \Psi(x,y,z,t)}{\delta y^2} + \frac{\delta^2 \Psi(x,y,z,t)}{\delta z^2}$", or $\nabla^2 \Psi(x, y, z, t)$) in the case of Hilbert arithmetic in a wide sense (i.e. the qubit Hilbert space), which is the essence of the Schrödinger equation. That idea about the fundamental arithmetical core of the Schrödinger equation needs much more detail for its rigorous proof being far beyond the scope and subject of the present paper. However, if it is granted ready as a fact, it allows for elucidating the relation of logicism and Hilbert arithmetic both referring to the foundations of mathematics:

As to any equation and thus to its particular case of the Schrödinger equation, logicism is able to erase (in virtue of its definition) the distinction between propositional logic (being the fundamental "zero-order" logic) and any first-order logic (i.e. any mathematical theory, including

[50] For example, as in the approach of Halbeisen, Saharon (1994) about the consequences of arithmetic for set theory, or Tzouvaras, A. (1992) about arithmetic and alternative set theories.

[51] The eventual extensionality of the qubit Hilbert space can be interpreted by the "generalized extensionality of fuzzy relations" suggested by Daňková (2004).

any physical theory as a mathematical formalism, which is quantum mechanics in the case at issue). So, any equation possessing a certain content in terms of its interpretation as a first-order logic (i.e. quantum mechanics) degenerates trivially to identity in the framework of propositional logic since the difference between the "left and right sides" of it is concentrated thoroughly in the additional axioms featuring the first order-logic and thus absolutely vanishing after restricting to the core axioms of propositional logic. So, the aforementioned many times above fundamental structure of a bit of information is able to describe exhaustively the relation of trivial identity in propositional logic to any of its meaningful interpretations as a certain first-order logic due to the complementing specific axioms and thus to the class of all first-order logics meant by set theory[52].

The suggested a little above idea of how the Schrödinger equation can be in interpreted only arithmetically, though in the generalized sense of Hilbert arithmetic, originates from set theory, i.e. it is to be related to *class* of all first-order logics rather than to its specificity of a certain first-order logic such as the mathematical formalism of quantum mechanics.

Speaking quite loosely, it means that a unit ("± 1") is the same being whether finite (e.g. in "$n + 1$" where "n" is a natural number or a finite ordinal number) or transfinite[53] (for example, as in "$\omega - 1$" where "ω" is a transfinite ordinal number): the same unit in either finiteness (conventionally, the "left side" of the Schrödinger equation) or infinity (in its "right side"). Following the same identity of a unit in both finiteness and infinity, one can equivalently interpret infinity as a second or dual finiteness just as Hilbert arithmetic does, introducing a dual and anti-isometric counterpart of Peano arithmetic.

Then, the meaning of the Schrödinger equation generalized to set theory and thus to any first-order logic (from where it originates supposedly) consists only in the identification of a unit ("± 1") referring to the class of all first-order logics in Peano arithmetic with the unit referring already to a certain first-order logic (whatever it be). So, the Schrödinger equation and the mathematical formalism of quantum mechanics, or quantum mechanics itself, to which it belongs, so meaning a certain first-order logic alone, but only at first glance, originates from (or can be generalized to) the class of all first-order logics meant by set theory. Indeed, the unit ("1") of the usual arithmetic can be obtained from the qubit Hilbert space as an "empty qubit" or as the class of equivalence of all values which can be "recorded" in an "empty qubit" therefore once again confirming the set-theoretical interpretation of the Schrödinger equation.

One might say more or less figuratively that the experiments of quantum mechanics can be related equally well to the fundamental mathematical and necessary properties or relation of the universe usually granted to be a physical entity rather than a mathematical one (as the present paper tends to conclude, though).

52 For example, Boyer, Lusk, McCune, Overbeek, Stickel. Wos (1986).

53 Especially, in the context of Gödel papers as in Woodin (2011) or Wigderson (2011).

III REINTERPRETING GÖDEL'S PAPERS (1930; 1931) AS ADVOCATING LOGICISM

The conceptual shift necessary for the interpretation Gödel's papers (1930; 1931) as advocating logicism (for example, against finitism and formalism[54] both understood rather in a strict or narrow sense) consists in realizing the negation of the conclusions of the former paper as a necessary condition for those of the latter paper therefore implying the mutual complementary of the Gödel completeness and incompleteness statements. The shift at issue is to be related to their usual interpretation of being simultaneously valid since both set theory and arithmetic are meant to be first-order logics and similar to any other mathematical theory also being first-order logic.

On the contrary, the algebraic approach as above to the triple of propositional logic, set theory, and arithmetic allows for revealing the unifying and thus underlying them structure of a bit of information though initially abandoning the distinction between zero-order and first-order logic[55], but immediately emphasizing that structure, and then secondarily and in its basis: restoring their distinction. Then, one can interpret the difference between "propositional logic" and "first-order logic" in an only formal way, which can be reduced to algebraic:

Indeed, there exists a core tuple of axioms identifiable as those of propositional logic referring to any propositions regardless of their content. One can add specific axioms to that, determining the content of propositions therefore indicating a certain first-order logic. If the class of equivalence of all possible first-order logics should be considered, the statement that the class at issue coincides necessarily with propositional logic after it has been defined as a list of axioms valid to the propositions at all seems to be obvious only in virtue of the definition of "class of equivalence". Nonetheless, the discussion is yet only intensional meaning all first-order logics and their class of equivalence.

Just set theory by its fundamental concept of set involves the extensional viewpoint as to the elements of any set defined in advance intensionally by its characteristic property also representable as a conjunction of a relevant list of axioms, Then, *actual infinity* being definitive and inherent to set theory can be meant by the whole of a set or a list of axioms fundamentally excluding the option for any infinite set to be exhaustively described by the description of all elements of it one by one resulting again (as in the former case of a proper infinite set) in a finite conjunction of "axioms" though now each axiom refers to a certain element.

So, one can notice that the distinction between extensionality and intensionality[56] is rather conventional always resulting into a finite collection of entities in the final analysis (also in the case of an *infinite* set but necessarily represented by its *finite* characteristic property), which can be interpreted both as propositions in the case of propositional logic (i.e., intensionally) and as

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[54] Logicism and formalism (as well as intuitionism) can be compared in relation to their ideas about the "three crises in mathematics" (e.g., Snapper 1979). On the contrary, Strauss (2011) reveals the convergence of Bernays's viewpoint and Gödel's reflections on the foundations of mathematics.

[55] It can be also revealed (or at least interpreted) about Husserl's "pure logic" (Isaac 2016; Hart 2004; Hanna 1984).

[56] For example, in the context of Widerker (1983) or Marcus (1960).

elements in the case of set theory (i.e. extensionally[57]). Emphasizing that that ultimate collection of entities is always finite in the final analysis, the only arithmetical description of it is to be sufficient *as if* involving and deducing a direct contradiction to the Gödel incompleteness paper.

Anyway, the following distinction between "finite set" and "infinite set" after logicism and starting from intensionality can be revealed[58]: the former can be defined both extensionally and intensionally as a "first-order logic" unlike the former, conserving the equivalence of extensionality and intensionality only as a second-order logic (to the tuple of axioms) being definable as a first-order logic only intensionally[59]. Then and as after logicism, the intentional description of any mathematical entity (including an actually infinite one) can be granted to be universal, only distinguishing first-order and second-order logics and postulating that there exist mathematical entities, to which the exhaustive or "complete" description in terms of any first-order logic is impossible and identifying them as being infinite or "actually" infinite according to the traditional notations and concepts of set theory.

Of course, the last statement is an additional axiom and only ostensibly inferred to be a theorem after the Gödel incompleteness paper (1931) meaning that the description of those logics (postulated to be only second-order) is necessarily incomplete in any first-order logic. One can immediately notice that the fundamental structure of a bit of information is quite sufficient for the intensional approach of logicism to mathematics able to involve "actual infinity" as those entities which are second-order logics fundamentally irreducible to any first-order logic as the Gödel incompleteness statement can be equivalently reformulated.

One can notice, that the incompleteness paper (1931) even reinterpreted as above only in terms of intensionality (after reducing extensionality to propositional logic, and infinity to second-order logics) contradicts Russell's original logicism at least in its initial forms as well, due to his "vicious-circle principle" as far as (or if) it is inconsistent with any higher-order (including second-order) logics[60].

Anyway, if logicism is understood in a wider sense, namely as reducing of the extensional viewpoints inherent to set theory to the intensionality of propositional logic, the Gödel incompleteness statement is consistent with it, needing only second-order logics, but not any higher-order logics by virtue of the Löwenheim-Skolem theorem and thus, in virtue of the axiom of choice in the final analysis.

Then, the necessary (for set theory in the thalweg of the generalized logicism) second-order logic can be reinterpreted to be complementary (and thus idempotent) to first-order logic therefore involving a meta-structure relevant to propositional logic itself rather than to arithmetic (as the hierarchy of types, for example): only alternating first and second-order logics, but avoiding any logics of order higher than two.

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[57] Hinnion (1986) investigates extensionality in the Zermelo-Fraenkel set theory, and Hinnion and Libert (2003) link it to intensionality.
[58] For example, Morrill (1990) though in the context of "boundedness".
[59] Urmson and Cohen (1968) discuss criteria of intensionality.
[60] For example, Jung (1999).

That approach corresponds to the pair of two dual Peano arithmetics in the framework of Hilbert arithmetic[61] allowing in turn for the reinterpretation of arithmetic only within the generalized logicism: speaking loosely, Peano arithmetic as the "half" of propositional logic so that it complemented by the other "half" is able to constitute the "whole" of propositional logic from an arithmetic viewpoint, or more generally said, an extensional viewpoint to intensionality, in which set theory takes the intermediate position between the only extensional arithmetic and the only intensional propositional logic, being simultaneously both extensional (for the concept of an "element of a set") and intensional (for the concept of the characteristic property of a set, which is a proposition). Thus, the concept of "set" itself serves to reconcile or equate the extensional viewpoint of arithmetic and the intensional viewpoint of propositional logic further defining the actual infinity inherent for set theory by the postulated option of the absolute identification of the two viewpoints.

Of course, the metaphor of arithmetic as the "half of propositional logic" so that one can obtain the whole of propositional logic (which one is to further identify with set theory) by adding the other half of arithmetics needs a rigorous definition. It can be achieved by the concept of a bit of information and the mediation of set theory (more precisely, by the concept of sets of bits and the axiom of choice[62]) as follows:

Propositional logic can be interpreted on a set of bits by the bijection of any proposition to a certain single bit so that its meaning either "true" or "false" to be related unambiguously to the two alternatives of the corresponding bit, and the binary operations of propositions, by means of the relevantly defined binary operations of corresponding bits. Then, the set of all bits corresponding to all propositions can be well-ordered utilizing the axiom of choice and that well-ordering to all bits divided into parts and also well-ordered, but separately already: the one well-ordering refers to the all "true" alternatives of all propositions; respectively, the other well-ordering will mean only all "false" alternatives of all propositions. Each of those two well orderings is able to satisfy the axioms of Peano arithmetic (including the axiom of induction).

Furthermore, the contradiction of the axiom of induction remaining valid to the so-defined two alternative Peano arithmetics to the axiom of infinity can be avoided: indeed, the axiom of infinity can refer only to the set of all bits (respectively, all propositions) consisting of two *finite* "halves"[63]: the one of all "true" alternatives constituting the one Peano arithmetics; and the other half of all "false" alternatives, or the dual counterpart of the former Peano arithmetic. Obviously, that definition of "natural number" can be thoroughly in the framework of logicism only by the mediation of set theory if the latter is identified with propositional logic in advance, for example:

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[61] Thus, "infinity" is interpreted as a "second finiteness" or as "finiteness at infinity" (Firby 1971) or following the "interpretability of arithmetic in set theory" (Collins, Halpern 1970). D'Alessandro (2018) suggests a more contemporary viewpoint.

[62] Kanovei and Lyubetsky (2012) equate the axiom of choice to the "pantachy existence theorem", relevant also to lattices, possibly Boolean in particular.

[63] One can compare that approach with the "logically simplest form of the infinity axiom" (Parlamento, Policriti 1988).

by sharing the same structure of Boolean algebra, Then, the metaphor of arithmetic as the "half of propositional logic" would be rigorously defined[64].

That construction of "bisecting propositional logic into two Peano arithmetics" needs the mediation of the set theoretical axiom of choice for the well-ordering of the set of all propositions respectively that of all bits. However, after the halving of the bits into "true" and "false" alternatives, one obtains two classes of all natural numbers in which all natural numbers are finite according to the axiom of induction. If they would be considered as sets, they would be again infinite, but this is not to be done:

So, one means an infinite set "before choice" (before the choice featuring any bit or the axiom of choice), but two finite Peano arithmetics "after choice", each of which is *incomplete* to the initial infinite set due to its dual counterpart of one more Peano arithmetic embodying the Gödel incompleteness statement as an axiom. The construction demonstrates the "obviousness" featuring any axiom, but nothing more: the construction cannot prove it rather emphasizing that it is an axiom.

The postulate of the nonstandard bijection of the two dual Peano arithmetics into a single one ($P^+ \otimes P^- \leftrightarrow P^0$) is equivalent to it. However both sides of it are finite and thus only in the framework of arithmetic, but the construction described above involves the set of all propositions or all bits postulated to be infinite in virtue of the axiom of infinite and this can be expressively emphasized by the following notation ($P^+ \otimes P^- \leftrightarrow \{P^0\}$) where the brackets "$\{P^0\}$" mean the set of all natural numbers or respectively, the Peano arithmetic "P^0", but only as a set notated to be just "$\{P^0\}$".

Furthermore, the "nonstandard bijection" if "$\{P^0\}$" is interpreted to be an infinite set according to the axiom of infinity in the framework of set theory needs the axiom of choice[65] in the same framework. However, the nonstandard bijection adds all Peano axioms as equivalent to any well-ordering, among which the axiom of induction is to be featured since it is transformed into the axiom of infinity by the newly introduced notation of the brackets; that is: "P^0" means the axiom of induction, but its notation by brackets "$\{P^0\}$" replaces it with its negation, i.e. the axiom of infinity. All the rest of Peano axioms remain the same in both cases since they serve to be axiomatically defined what "well-ordering" means.

One may use the visualization by the pair of Euclidean and non-Euclidean geometries as it is discussed in much more detail in the previous *Part I* of the paper just in relation to the pair at issue (that of arithmetic and set theory). Then, the notation by brackets now interpreted to some Riemannian manifold would correspond to the transition from a local (i.e. in an infinitesimally small neighborhood about any point) description, to which Euclidean geometry is relevant, into a

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[64] The same metaphor in relation to infinity and in the context of Hegel's dialectics or dialectical logic can be revealed in Usó-Doménech,. Selva, Antonio, M. B. Requena, Segura-Abad (2017); Usó-Doménech, Selva, Requena (2016), or even in Ushenko (1949); and Posy (2008) links infinity to Kant's intuition. The logic of infinity is the subject of Sheppard (2011). On the contrary, Borkowski (1958) considers the "reduction of arithmetic to logic based on the theory of types without the axiom of infinity".

[65] Germansky (1961) considers the axiom of induction (i.e., in the framework of arithmetic) and the axiom of choice (i.e., the framework of set theory) jointly.

global description (i.e. in a finite neighborhood about any point), to which non-Euclidean geometry is relevant[66]:

The identification of both states (speaking loosely that "before brackets" with that "after brackets") features all "gauge" theories in physics and particularly the Standard model sharing with the nonstandard bijection (as here) the same conceptual origin from quantum mechanics. Then, one might say that only gauge theories would be relevant to the nonstandard bijection or to the eventual origin of arithmetic from propositional logic as its "half" where the concept of set and more generally, set theory is the necessary mediation able to introduce a relevant "gauge symmetry".

Further, the suggested construction of arithmetic by propositional logic can be equivalently inferred by the model of a Turing machine[67]: to the class of the interpretations of which any contemporary computer belongs; and the construction itself reflects the well-known fact that a Turing machine is able to accomplish both arithmetical and logical operations equally well, including mixed in any way within a correct algorithm.

Quantum computer can be represented as a generalized Turing machine processing an actually infinite calculation so that it would not ever end, being accomplished in a Turing machine defined standardly (e.g., Penchev 2020 July 21). However, a quantum computer, following the ideas above, can be represented by two Turing machines. Of course, the couple of any two independent Turing machines cannot be thought to be a quantum computer: they need a relevant additional condition to be considerable as a quantum computer:

For example, the other Turing machine (which can be conditionally notated as "TM2") should fulfill a relevant meta-algorithm in relation to that realized by the former Turing machine (which can be conditionally notated as "TM1"). A visualization of that correlating processing by both TM1 and TM2 could be the following: TM2 calculates the probabilities corresponding to parallel branches of a certain quantum algorithm, and TM1 continues to calculate in that branch, the probability of which is currently maximal. If its probability as a result of the work of TM1 ceases to be the maximal probability, TM2 starts or continues the calculation according to the new branch with the maximal current probability, i.e.: within it.

Anyway, that correlation of TM1 and TM2 (in fact implementable practically by a relevant architecture of a single real computer) is not sufficient theoretically for their pair to be a model of quantum computer rather than an approximation to it. TM1 and TM2, subordinated rigidly so that TM2 to be in a meta-position to TM1, would need a series of TM3, TM4, … TM*n*, …, each of which would be in a meta-position to the previous one, i.e., continuable *ad lib*. Restricting it to a

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[66] One can compare this with the link suggested by Oleinik (1994): the "connection of the classical and quantum mechanical completeness of a potential at infinity on complete Riemannian manifolds".

[67] Makowsky (2008) discusses "logic for a computer scientist" in the context of Hilbert's program. Lambda calculus suggests its own way for relating logic to extensionality (Intrigila, Statman 2005; Hindley, Longo 1980).

certain natural number, only an approximation[68] of a quantum computer of an order "*n*" would be defined and thus more or less relevant for realizing the meant quantum algorithm.

That series *ad lib* implying always to be not more than an approximation can be anyway avoided by complementing at least theoretically TM1 to be in a meta-position to the calculation of TM2 simultaneously (as well), so that they to be idempotent to each other in an exact similarity to any two conjugate quantities in quantum mechanics also representable by a relevant gauge symmetry. However, the problem about the implementation by the architecture of a real computer (and thus a Turing machine) seems to be very difficult, fortunately, out of the scope of the present paper.

Meaning the approach sketched above only as to the incompleteness paper (1931), one can continue it as to the former, completeness paper (1930) in a way to reinterpret it to advocate logicism regardless of Gödel's real intention more or less close or far to that objective. Two theorems, "Satz VII" and "Satz X" would remain the key ones just as in the case of the standard interpretation. The former can be meant as the "completeness theorem", and the latter, as the "compactness theorem".

The completeness theorem would refer to the relation of propositional logic to set theory in order to identify them as two interpretations of the same structure of Boolean algebra, and the compactness theorem means the relation of the structure at issue to arithmetic. The latter relation is elucidated above by the rigorous construction corresponding to the metaphor for arithmetic to be the "half of propositional logic" by the mediation of the concept of a bit of information and the axiom of choice (implying that of "set", respectively set theory, in relation to all propositions or all bits) so that if arithmetic can be complemented by a dual counterpart of Peano arithmetic (as in Hilbert arithmetic), their pair can be isomorphic (or algebraically homomorphic) to set theory.

The meaning of both completeness and compactness theorems can be easily represented by means of Hilbert arithmetic and quite relevantly to the intention of reinterpreting them as advocating logicism though in a rather generalized sense[69]. The former means homomorphism of propositional logic and set theory as the same structure of Boolean algebra. Thus, the mutual completeness of propositional logic and set theory to the other one is trivial since they are the same mathematically.

Furthermore, both theorems are equivalent to each other, but the latter means the relation of set theory to arithmetic in a way opposite to its consideration in the incompleteness paper (1931). That approach would be absolutely consistent if the viewpoints of the completeness paper (1930) and the incompleteness paper are granted to be complementary to each other (as in the present study). This means: the latter paper is consistent with the negation of the conclusion of the former paper about the relation of arithmetic and set theory, but in a consistent way, i.e. elegantly avoiding any direct contradiction as in the case if they are meant anyway simultaneously.

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[68] That inherently necessary approximation can be considered in the context of philosophical interpretations of the Gödel incompleteness theorems (e.g., as Benacerraf 1967).

[69] For example, as in Zach (2003).

Indeed, the compactness theorem means all finite *sets* relevant to finite models rather than arithmetic directly. The difference consists in the context: the concept of "finite set" unlike that of a class of natural numbers admits infinite sets. The difference can be represented especially discernibly and distinctly by the two dual copies of Peano arithmetic, featuring Hilbert arithmetic. All finite sets in its framework consist of the Cartesian product of all finite sets in each of both dual Peano arithmetics. On the contrary, the class of all natural numbers in Peano arithmetic if it is considered as a set would mean only the one of them therefore being inherently incomplete to the Cartesian product at issue.

In other words, or speaking loosely, all finite sets refer to both "halves of propositional logic" (a.k.a. "set theory" being homeomorphic to propositional logic according to the completeness theorem) unlike the class of all natural numbers meant in the incompleteness paper since it relates only to the "one half" of it. So, the direct contradiction between the conclusion of the compactness theorem to that of the incompleteness theorem is really avoided since they can be consistently distinguished as referencing to different mathematical entities though very similar, at least at first glance:

The compactness theorem means both "halves" of propositional logic after the completeness theorem identifies it with set theory and thus to all finite sets relevant to finite models. On the contrary, the incompleteness theorem refers only to the *one half* since it indicates the class of all natural numbers being relevant to arithmetic. So, the set of all subsets of the set of all natural numbers relevant to the incompleteness theorem is a *true* subset of the set of all finite arithmetic sets. Thus, the compactness theorem means a certain set to which the set meant by the incompleteness theorem is a true subset. They refer to different mathematical entities, which allows for them to deduce a statement and its negation (correspondingly, completeness and incompleteness) to what each of them is to be related.

If one considers the relation of the two theorems (ostensibly contradicting each other, but only at first glance) in terms of a bit of information, it can be visualized as follows. The incompleteness theorem refers to the relation of the chosen alternative to the bit as a whole: the incompleteness proved in it is due to the complement of the unselected alternative to the bit as a whole; or in other words, it refers to the state of a bit after the choice of either alternative. On the contrary, the compactness theorem means the state before the choice of either alternative, therefore being definitively complete.

Furthermore, one can identify the opposition of the compactness and incompleteness theorems with the opposition of finiteness and infinity by virtue of the fundamental complementarity of the two oppositions of a bit of information: (1) "before choice" versus "after choice"; (2) the one alternative versus the other alternative in the latter case. Since those two oppositions are complementary, one can postulate them to be identical (as well as, not to be identical because the check of identity requires for them to be simultaneously available)[70].

As an interpretation of the above structure of a bit of information, the opposition of the compactness and incompleteness theorems can be postulated to be the same as that of infinity (for

[70] Hofstede and Weide (1998) derive "identity from extensionality".

the compactness theorem) versus finiteness (for the incompleteness theorem). After introducing their interpretation by the two dual Peano arithmetics of Hilbert arithmetic, the compactness theorem corresponds to the pair of the two Peano arithmetics, and the incompleteness theorem, to a single one, therefore exemplifying the nonstandard bijection by a homomorphism of Hilbert arithmetic and Peano arithmetic, on the one hand, as well as by means of the correspondence of the two alternatives of a bit "after choice" to their coherent and indistinguishable correlative state "before choice", on the other hand.

So, the nonstandard bijection itself may be considered to be an additional axiom corresponding to the identification of the two oppositions of a bit of information being consistent because of their complementarity. Indeed, anybody may alternatively reject it in a not less consistent way. The Gödel incompleteness statement (after the consideration to be an independent axiom or meta-axiom in the previous *Part I*) corresponds unambiguously to accepting the postulate of the nonstandard bijection.

Analogically, it can be refused not less consistently therefore postulating its negation, for example introducing the quantity of the "distance between finiteness and infinity" (as in *Part I*) including the extremal particular case of the absolute coincidence of finiteness and infinity furthermore identifiable with postulate of the nonstandard bijection (respectively, with that of the Gödel incompleteness statement) in the a way similar to that for Euclidean geometry to be considered as a particular case of non-Euclidean geometry for the constant parameter of zero space curvature.

If one returns to the biblical metaphor for the "expulsion from Paradise", it would correspond to the former opposition of a bit of information, namely that of the states "before choice" (within Paradise) versus "after choice" (out of Paradise). The state "after choice" consists only in the distinction of "evil" and "good" and can be granted to be identical with the former opposition where "good" would correspond to the state "within Paradise", and accordingly "evil", to that "out of Paradise".

One can again assure that the Gödel incompleteness statement corresponds exactly to the Old Testament myth in the interpretation of Judaism rejecting any option about the "Redemption of the original sin": the good of being within Paradise is lost forever and thus irrecoverable in any way. On the contrary, granting the Gödel incompleteness statement to be an axiom and hence admitting the option of its negation, one moves in the thalweg of the New Testament and Christianity: one can atone for the original sin even after the expulsion of Paradise though by virtue of God's help by Christ, the Redeemer: the concept of Hilbert arithmetic (borrowed from quantum mechanics and its completeness, in fact) serves for that "redemption of the original sin" of arithmetic due to its inherent incompleteness[71].

Anyway, one can realize logicism as well as both papers of Gödel as its apology since arithmetic due to its incompleteness is an "expulsion" from the paradise of propositional logic,

[71] One can trace incompleteness even to "reason choice" (Sen 2004) in the same chain linking incompleteness, choice, the axiom of choice, and well-ordering to arithmetic in the final analysis.

which did not know the “original sin” of the distinction between infinity and finiteness after Cantor’s set theory and Peano arithmetic[72].

IV THE PAIR OF ARITHMETIC AND SET THEORY COMPLEMENTED BY PROPOSITIONAL LOGIC; THE GÖDEL COMPLETENESS PAPER (1930)

Two main statements to be demonstrated are the subject of the present section:

(1) Both set theory and propositional logic can be exhaustively represented as two different interpretations of Boolean algebra as long as set theory deliberated from the hierarchy of infinities (established by Cantor rather in virtue of metaphysical considerations than because of any proper mathematical necessity) being equivalently substitutable by two idempotent (or dual, or complementary) entities such finiteness and infinity, which is also the most natural suggestion.

Then, Gödel’s “completeness paper” (1930) can be rather reinterpreted as a pathway to the completeness of set theory to propositional logic (unlike it to arithmetic claimed to be either incomplete or inconsistent according to the next paper in 1931, being an almost trivial observation if both are equated to be different interpretations of the same mathematical structure (usually granted to be algebraic). Indeed, any two or more interpretations of the same structure are trivially complete to each other: otherwise, they might not share the same structure absolutely.

There exists one more and very essential difference between propositional logic and set theory even sharing the same structure, but being two interpretations in different hierarchical levels since set theory is a first-order logic to which propositional logic itself can be qualified as a “zero-order” logic, i.e. referring to all propositions of set theory (whether true or false, or eventually insoluble if those can be proved to exist in set theory alone) only as propositions independent of their proper contain studied properly by set theory. That circumstance is to be related to the shared single structure of both: Boolean algebra.

(2) Boolean algebra can be split into two dual anti-isometric Peano arithmetics and thus identifiable as a whole to be Hilbert arithmetic in a narrow sense. Peano arithmetic just as set theory is usually granted to be a first order-logic, however it unlike set theory is only a “half” of the structure shared by propositional logic as the relevant “zero-order logic” referring to all arithmetic propositions only as to abstract propositions independent of their proper arithmetic contain.

Both statements, (1) and (2) above, involve implicitly the concept of information, on the one hand, as far as they mean the elementary structure of a bit of information though interpreted differently in virtue of the additional consideration in each of the two statements. On the other hand, one can claim not less relevantly that the aforementioned “nonstandard bijection” can link the two statements. A few notices can be useful to elucidate additionally the sense of both statements:

The identification of the concepts of ‘proposition’ and ‘set’ is a necessary condition for the identification of set theory and propositional logic as two interpretations of Boolean algebra. It had

[72] The debates about infinity in mathematics at the end of the 19th century (Laugwitz 2002) can be instructive.

been meant by Whitehead and Russell as a fundamental idea in the foundations of mathematics according to logicism or *Principia mathematica*. In the final analysis, set theory seems to be a theory of propositions meant by their extensions of elements of sets to which they refer, but essentially complemented by the notion of actual infinity and Cantor's hierarchy of infinities, which they (or least Russell 1908; 1907; etc.) intended to replace by the theory of types[73] (also relatable to propositions) and furthermore, deliberated from the paradoxes of set theory just prohibiting self-predicativeness as their general reason.

In other words and less formally, the concept of "set" can be understood in both ways: (1) as a whole by its characteristic property, which is a proposition and all elements of it obey (as well as all entities which are not elements of that set do not satisfy it), resulting furthermore in the interpretation of Boolean algebra as propositional logic[74]; (2) as all elements of it, which is the proper interpretation of set theory and can be distinguished from the former only by the additional opposition (Cantor's proper innovation): infinity (i.e. the axiom of infinity in set theory[75]) versus finitenes (which is equivalent to the axiom of induction)[76].

Then, set theory is intuitively understood as a theory of both finite and infinite sets versus arithmetic traditionally understood to be the theory of all natural numbers which are not identified with all finite sets though the equivalence of arithmetic to the theory of all finite sets can be easily inferred since all finite ordinal numbers and all natural numbers are the same[77].

So, the two oppositions inherent for the definition of a bit, namely: (1) "before choice" versus "after choice" (of either alternative of a bit); (2) the explicit choice of the one alternative of a bit versus the other one, can be immediately revealed in the triple: propositional logic (before choice of "either arithmetic or set theory") versus the other opposition after choice and consisting in the explicit opposition of "either arithmetic or set theory".

As far as set theory is usually and traditionally interpreted to be a theory of both finite and infinite sets, the fact that Boolean algebra is the shared structure of propositional logic and set theory means that the two alternatives, all finite sets (i.e. arithmetic) versus all infinite sets distinguishable only by the contradiction of the axiom of induction versus the axiom of infinity, as a whole and repeating the state "after choice" as in turn unifiable with the state "before choice" meant by propositional logic since any proposition suggests a kind of "epoché" whether it refers to an infinite or to a finite set as its characteristic property.

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[73] Russell's theory of types (e.g., Consuegra 1989; Cocchiarella 1980) met criticism including that by Wittgenstein (e.g. Han, DePaul 2013; Rufino 1994; Davant 1975), but a class of contemporary theories of types originated from it (e.g. Fox, Lappin 2015; Copi 1971). The theory of types was discussed by Gödel in 1939 (Cassou-Nogues 2009).

[74] For example, Faust (1982).

[75] Blass (1989) discusses the axiom of infinity linked to the category of set. Pambuccian and Struve (2020) investigate infinity in the foundations of geometry as well as Gordin (1919). Hilbert arithmetic discussed also in the present paper equates infinity in geometry (by the mediation of Hilbert space) and that meant by set theory and opposed to finiteness inherefent for arithmetic.

[76] For example, Hochberg (1977) links the axiom of infinity to properties and abstracts.

[77] The concept of finite ordinal numbers is consistent with extensionality, nonetheless Dzierzgowski (1998) suggests an intuitionist alternative "without extensionality".

In other words, propositional logic can be thoroughly identified with the concept of "Fermat arithmetic" introduced in another paper (Penchev 2021 March 9) in order to distinguish the inexperienced naivety of Fermat's age from Peano arithmetic (particularly meant by Gödel) after the explicit Cantorian set theory in our own epoch. However, that identification can seem paradoxical as far as Fermat arithmetic should ostensibly mean a *single* Peano arithmetic contradicting directly the two dual Peano arithmetic able to constitute Hilbert arithmetic in a narrow sense: not at all, Fermat arithmetic, residing in blissful ignorance in relation to infinity, therefore extends that ignorance to the problem "either a single or two dual Peano arithmetics" which in turn can result into the eventual indistinguishability of Fermat arithmetic from propositional logic in the final analysis. This means after translating it into the language of contemporary ideas that Fermat arithmetic as a first-order logic to propositional logic might be identifiable with the latter as Boolean algebra just as set theory nowadays.

However, Peano arithmetic being already "expelled from Eden" after humankind "knew the sin of actual infinity" by Cantor's set theory cannot be more identified with propositional logic as the same structure of Boolean algebra (now admissible only for set theory itself) being incomplete (more precisely, either incomplete or contradictory) to it just as a corollary from the literal Gödel dichotomy of the relation of arithmetic to set theory and shared by contemporary mathematics at all since it knew the same "sin of actual infinity" and called "Gödel mathematics" to be distinguished from the "happy" mathematics in Eden (i.e. which "did not yet know the original sin") and called "Hilbert mathematics".

Another necessary condition for the identification of set theory and propositional logic as two interpretations of Boolean algebra consists in the equivalence of Cantor's hierarchy of infinities and their duality (or idempotency, or complementarity) reducing it to two options similar to those of logical negation. The basis is the axiom of choice implying further the Löwenheim - Skolem theorem discussed already above in the present context. Anyway, it does not mean literally the same kind of idempotency as logical negation but the reduction of Cantor's hierarchy of infinities to a single kind of infinity: countable infinity.

Then, the difference in comparison with idempotency interpreted literally would consist in the fact that (for example) continuum should be identified with countable infinity rather than with finiteness as idempotency needs. So, the Löwenheim - Skolem theorem is to be essentially complemented by Skolem's "relativity of the concept of 'set'" including even finite sets as he expressly emphasized in his presentation in 1922 and inferred in the present paper as an explicit construction relying on the Dedekind-like (proper set-theoretical finiteness) and the correspondence between Hilbert arithmetic in a wide sense and Hilbert arithmetic in a narrow sense:

That approach can be featured by a few main stages or properties: (1) the understanding of two successive infinities belonging to Cantor's hierarchy by means of their relation similar to that between finiteness and infinity; (2) the equivalence of that uniform relation to a gap between two successive dimensions (particularly that between two successive dimensions of Hilbert space), but not less relevantly, between it and its dual space being inherently idempotent to each other; one has to emphasize that the equivalence at issue identifies the *hierarchical* gap between two successive dimensions of Hilbert space with that between its two dual spaces therefore needing a relevant Hilbert space, for example the qubit Hilbert space of Hilbert arithmetic, as a carrier of the fundamental equivalence of hierarchy and idempotency (duality); (3) the understanding of the transition from infinity to finiteness as a kind of projection corresponding to the gradual "narrowing" of the corresponding probability density distribution shared also by the "collapse of wave function' in the process of decoherence due to quantum measurement (e.g. Penchev 2022 August 3); (4) the interpretation of the relation between any two successive Cantorian infinities uniformly by the same idempotent structure, in fact, borrowed from Hilbert space (particularly from the qubit Hilbert space).

Finally, one can notice that the necessary equivalence of idempotency and hierarchy is exhaustively represented in Hilbert arithmetic in a narrow sense though its deduction and justification turn out to be hidden in Hilbert arthritic in a wide sense (as above). Thus, the way of Whitehead and Russell in *Principia mathematic*a to "get rid" of Cantor's hierarchy of infinities in favor of logicism only by means of the theory of types and not utilizing any elements of Hilbert arithmetic whether in a wide or in a narrow sense is especially interesting.

In fact, they restricted themselves to replace the sense of hierarchy substituting Cantor's initial hierarchy of infinities being inherently extensional, i.e., referring to all elements of infinite sets, with the propositional hierarchy of types being, on the contrary, intensional since it means the corresponding propositions, which all elements of the infinite sets obey.

Consequently, the link between hierarchy and idempotency, being an explicit subject of research after Hilbert arithmetic has been involved, remains implicit and unarticulated in their approach hidden in the concept of "proposition": on the one hand, it can obey the structure of Boolean algebra (being idempotent due to the operation of logical negation); on the other hand, the collection of propositions can obey Russell's hierarchy of types postulating an analogical dimensional gap by virtue of Russell's rule of "vicious circle".

Anyway, the inherently probabilistic essence of the transition over the gap, furthermore absolutely necessary for justifying the equivalency of idempotency and hierarchy in detail, cannot be reached so: because it needs the fundamentally newly worldview of quantum mechanics, by the by, appeared and, first of all, established a few decades later than the initial publication of *Principia mathematica*.

As a general though rather technical conclusion, Hilbert arithmetic in a wide sense can remain thoroughly hidden for the former of the two statements (which are the subject of the present section) to be inferred since its narrow sense is absolutely sufficient for that objective.

The premises are borrowed from Whitehead and Russell's "logicism": (1) any set is equivalent and thus can be equivalently substituted by a certain proposition, which is the characteristic property of the set and able to distinguish unambiguously its elements from any other entities; then, set theory can be thoroughly investigated as an interpretation of propositional logic therefore sharing the same structure of Boolean algebra; (2) the type theory utilized in Russell's type theory also is necessary to be involved in order to justify the equivalence of idempotence valid as to all propositions in the following sense:

Any proposition implies its unambiguous and idempotent counterpart in virtue of the operation of logical negation. The theory of types establishes a well-ordering to the order of all propositions prohibiting any self-predictability by a special postulate known as Russell's "vicious circle principle"[78]. Then, both Cantor's hierarchy of infinities (also known as the hierarchy of "alefs" notating types of cardinal numbers of infinite sets) and Russell's type of propositions are underlain by the same fundamental structure of all natural numbers.

This gives the option that one can consider a special kind of type propositions defined by the condition that any "alef" is to correspond unambiguously to a certain level of type. That bijection of types and alefs being unambiguously and arithmetically enumerated by all natural numbers does not need either the "set of all types" or the "set of all alefs" both being compromised by paradoxes similar to the initial one suggested by Russell (1902) in a letter to Frege[79]. For example, the bijection at issue can be built as follows:

The proposition of any level means an infinite set of elements and then, the proposition relevant to any next level means the set of all subsets of the set of the previous level. An axiom of (ZFC) set theory guarantees that the set of the next level being so defined always exists. Furthermore, the Löwenheim - Skolem theorem allows for all infinite "alefs" to be identified in virtue of its premises, and now, that identification to be extended to all corresponding "type propositions" due to the bijection at issue.

Anyway, the abyss between any finite sets and their derivative sets consisting of all sets of the former sets being necessarily also finite, on the one hand, and any infinite set, on the other hand, is insurmountable in the framework of the suggested construction: finiteness and infinity, respectively, the classes of infinite sets versus those finite sets[80], are to be declared as idempotent similar to the pair of corresponding propositions (that a certain set is finite and its negation, that the same set is infinite, i.e. it is not finite). In the present context based on the opposition of intensional and extensional aspects, the same structure initially borrowed from the interpretation of Boolean algebra as propositional logic allows for any individualization of elements belonging to the set meant by the proposition being its characteristic property.

Therefore, though Hilbert arithmetic is much richer theory able to describe explicitly and constructively, however probabilistically, the mutual transition between finiteness and infinity, that description can be "bracketed" so that one needs only the fact that all "infinities of Cantor" can be effectively reduced to a single infinity, and then, opposed to finiteness in turn being irreducible to infinity in that way therefore implicitly suggesting their idempotency, duality, or complementarity (according to) the area of interpretation.

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[78] For example, Rouilhan (1992), Hilton (1992), Fleischhacker 1979), or linked to the "linguistic hierarchy" (Pap 1954), and Chihara (1978), discuss the vicious circle principle in relation to ontology.

[79] Also in relation to the present context: Hale (2005).

[80] Or the general relation of "extensionality, attributes, and classes" (Pap 1958).

Further, propositional logic and set theory as a first-order logic though both eventually sharing the same structure of Boolean algebra should be distinguished according to the kind or scope of propositions admissible in each of them and thus what follows if one applies Boolean algebra as set theory unlike its application as propositional logic.

In other words, adding the general concept of elements, set theory as a first-order logic is already able to discuss and consistently introduce new concepts irrelevant in the framework of propositional logic such as quantifiers or relations referring to certain elements meant by the corresponding set only as all elements. Obviously, the newly determined elements by quantifiers or relation can be individualized in turn as a set and characteristic property by a corresponding proposition also but indistinguishably obeying Boolean algebra.

Thus rigorously and mathematically, propositional logic and set theory (after they have been identified to be both Boolean algebra) can be anyway distinguished as the class of all structures of Boolean algebra versus the case to be distinguished from each other because of their different interpretations of the same Boolean algebra. That is: set theory means the class of interpretations of Boolean algebra versus the class of all structures of Boolean algebra.

If one utilizes the metaphor of the opposition of mathematics versus the world, the pair of propositional logic and set theory builds an image of the same opposition, but now within the proper framework of mathematics itself so that propositional logic is to be doubled by set theory distinguished from the former only postulating for it to be a first-order logic unlike propositional logic not being a first-order logic since it does not admit any extensional consideration. Then, one can conclude that set theory means the class of all first-order logics versus propositional logic generalizing their corresponding mathematical structures of Boolean algebra as a class.

A similar argument elucidates that the way of doubling propositional logic by set theory implies immediately and obviously for the former to be complete to the latter: what a looser interpretation of the one ("Satz VII")[81] of the main results of Gödel's PhD thesis or paper (1930) might be. Following the fundamental distinction between propositional logic and set theory interpreted as the class of equivalence of all first-order logics (loosely speaking, each of which refers to a certain set additionally specified as a *relevant compose proposition* being the subject of the theory at issue[82]), that theorem states that any formula belonging to propositional logic is provable in set theory (as well as there does not exist any statement provable in set theory, which is not simultaneously a valid formula of propositional logic).

If both propositional logic and set theory are two interpretations of Boolean algebra furthermore not possessing any additional features able to distinguish them from each other, that theorem of Gödel (1930, "Satz VII") seems to be an obvious and even trivial statement. Propositional logic and set theory differ from each other only by the assignability of a certain (i.e.

[81] The other main result is the compactness theorem ("Satz X") also relevant in the present context, discussed above and equivalent to the completeness theorem in Gödel's paper (1930) itself as well as after the consideration here.

[82] Following the same idea, set theory can be simultaneously considered to be a certain theory among the class serving for its definition postulating the absence of any additional specifications as its "relevant compose proposition".

unambiguous) relevant extension called "set" of "elements" to any proposition available in both cases. So, propositional logic means all propositions thus obeying Boolean algebra, and set theory means just the same propositions only doubled by relevant sets therefore also obeying Boolean algebra and nothing else since any proposition is equivalent to the pair of itself and its relevant set, to which the proposition at issue is the characteristic property.

A relevant transition to the latter statement, which is the other subject of the present section (namely, the following statement above: "Boolean algebra can be split into ywo dual anti-isometric Peano arithmetics and thus identifiable as Hilbert arithmetic in a narrow sense"), is the compactness theorem in Gödel's paper (1930, "Satz X"). It states that the necessary and sufficient condition for any infinite set of formulas of propositional logic to be valid is any finite subset of it to be valid.

One should analyze that statement meaning specific restrictions of the present context: only after "Satz VII" has established the equivalent mutual changeability of formulas of propositional logics and their unambiguous counterparts referring to sets and eventually, to their elements, "Satz X" suggests the formulas of propositional logic to be investigated as elements of relevant sets in virtue of the already proved "Satz VII". Meaning just that research, one is to emphasize expressly that the pair of "finiteness" and "infinity" can be related only to sets and extensions, but not to propositions directly.

Returning to the discussed structure of a bit of information being formed by two complementary oppositions, "Satz VII" and "Satz X" in question can be also situated in an analogical framework of two complementary oppositions. Then, "Satz VII" means the initial opposition (notated above to be "before choice" and "after choice" between the two alternatives of any bit) where the theorem establishes the fundamental equivalence of "intensionality" (meant by the class of all propositions[83] relevant to propositional logic) and itself, but already unambiguously doubled by "extensionality" (i.e. any proposition relevant to "intensionality" is doubled by its counterpart of a certain set consisting of elements even being zero as in an empty set, such that the proposition at issue is its characteristic property). Then and once "extensionality" has been involved in "Satz VI", itself in turn can be divided into "finiteness" and "infinity" therefore constituting the latter complementary opposition to the former opposition. Properly, "Satz X" states that the distinction meant by the latter opposition is irrelevant to the former opposition: a theorem proved by Gödel just to the particular case meant in his paper, but in fact originating and inferable from the general structure of a bit of information.

Furthermore, one can add to the above observation also the distinction of the multiplicative axiom (utilized by Whitehead and Russell in *Principia mathematica*) and the axiom of choice granted to be an exact equivalent (i.e.: neither weaker, nor stronger) of the latter, especially in relation to the equivalence to the well-ordering "theorem". The distinction consists in the opposition of the directions of their statements in a sense:

The axiom of choice suggests as its premise that any infinite set (since the choice from any *finite* set can be made without its assistance), from which a binary choice between the set consisting

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[83] Marti (1993) researches for the source of intensionality.

of any element and its set-theoretical complement is always possible. In other words, any subset of an infinite set and its complement to the infinite set in question does not contain any shared element as well as no element of the infinite set belonging either to the subset or to its complement therefore literally generalizing the analogical statements to finite sets:

Then, one can see the axiom of choice otherwise before that, noticing that it makes sense only extensionally, only to elements of any (infinite) set, only to any first-order logic, the class of which is identified with set theory itself rather than to propositional logic thus being acceptable as to Russell and Whitehead's logicism, for which extensionality including that of any infinite set has to be derivative from propositional logic and thus secondary to it.

Meaning that attitude of logicism, one can notice that the axiom of choice as to all propositions properly (i.e. regardless of any set of elements assignable to it in set theory unambiguously) establishes that any proposition is the same regardless of whether determines a finite or an infinite set consequently restricting the opposition of finiteness versus infinity to make sense only to extensionaly being indistinguishable in the framework of intensionality, which in turn is only meant by propositional logic[84].

If one utilizes again the metaphor that set theory as the class of all first-order logics serves to build (or even only to transfer) the world external to mathematics within itself, the sense of axiom of choice can be interpreted as the relevance of the concept of infinity (or more precisely, that of an infinite set only to the world and the derivatively and secondarily to mathematics itself in the area of set theory where an image of the external world is to be built).

The idea of logicism can be seen to be opposite to that of an image of the world external to it to be defined only within itself by set theory: namely set theory to be inferred from propositional logic; at that, rigorously formally and mathematically. Of course, that eventual deduction of set theory inferable from propositional logic alone means a further objective: a proposal of how the world can be inferred from mathematics at least in a looser manner acceptable for philosophy.

One of the main obstacles (if not the single one) is the concept of "actual infinity" (respectively, "infinite set" following formally and logically the axioms of set theory). The problem is that: infinity to be now inferred from propositional logic avoiding its postulation in set theory after Russell's paradox[85] (and many others after it) can demonstrate serious troubles about "set" or "infinite set".

Needless to say that common sense's philosophy embodied in the modern organization of cognition felt itself challenged by the wider philosophical ideas of *Principia mathematica* tending to be the "principles of the world" therefore rebelling to the episteme. Common sense's philosophy needs set theory and actual infinity in particular to be inconclusive from propositional logic and accordingly, Whitehead and Russell's undertaking to failure.

On the contrary and in relation to the particular problem about the axiom of choice, Russell suggested that the "multiplicative axiom" is able to replace the former, furthermore elementarily deducing the multiplicative axiom from the well-ordering theorem usually interpreted as a proof

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[84] For example, Lithown and Marras (1974) discuss propositions "without extensionality".

[85] For example, in relation to Russell's "vicious-circle principle" (Vardy 1979).

of the equivalence of the axiom of choice and well-ordering theorem (in fact, only by the mediation of the multiplicative axiom).

Meaning the general intention of logicism, one can reinterpret the multiplicative axiom as follows. Any two propositions can be synthesized meaningfully (or formally consistently) so that the Cartesian product of their corresponding sets (to which each of them is the characteristic property) is not an empty set (as far as an empty set is the extensional correspondence of both inconsistency and nonsense). In other words, the multiplicative axiom literally formulated is the extensional counterpart of an intensional statement (thus referring to propositions regardless of their extension or sets whether finite or infinite in set theory) penetrating Western philosophy and expressly discussed a long time ago e.g. by Hegel[86] or even still by the philosophers of Hellas, by medieval logicians up to the present day again and again though embedded in new forms: so, that is quite not a newly introduced invention of logicism.

That "any two propositions can be synthesized meaningfully" can be generalized but simultaneously specified additionally if one states that 'any two propositions are able to constitute a joint metaphor, i.e. to be linked metaphorically'. Then, the former statement (being furthermore the intensional counterpart of the multiplicative axiom) can be formally inferred from the latter by Gentzen's cut rule allowing for any metaphor[87] to be reduced to a single proposition (demonstrated for example in: *Penchev 2022 August 3*), otherwise crystallizing or appearing as a residue in the prolonged use of any metaphor.

Now, one can notice that the axiom of choice being properly and only extensional since it refers to the elements of any infinite set can be both identified with the multiplicative axiom and distinguished from it, first of all, in virtue of its intensional counterpart, with which it is equipped unlike the axiom of choice itself. The difference consists in the distinction of propositional logic from the class of all first-order logics (as set theory can be interpreted). Then the axiom of choice is to be related only to set theory and by it, to any first-order logic further (even to propositional logic if it is interpreted, by the by, in a quite admissible way, as a first-order logic: e.g., in the compactness theorem translating the universality of propositional logic in terms of finite or infinite sets featuring set theory or any first-order logic), and the multiplicative axiom, to both propositional logic and set theory.

Thus, the "multiplicative axiom" is designed to propagate the intention and conception of logicism that set theory is reducible to propositional logic after any set is reducible to its intensional counterpart of a corresponding proposition since the distinction between finiteness and infinity, though being fundamental for set theory itself, is, in fact, insignificant to its foundations and justification. One is to emphasize that closeness to finitism, constructivism or Hilbert program especially in the context of both papers of Gödel (1930; 1931) rather advocated the completeness of logicism versus the incompleteness of the enumerated schools originating from arithmetic or at least inherently linked to it:

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[86] For example, Masciarelli (2000) means the shared context of Russell and Hegel.

[87] "Metaphor" can be interpreted as 'context change' or 'indexicals' and then, they can be linked to intensionality as Chierchia (1994) or Forbes (1987) do.

Those schools grant the extensional viewpoint of set theory and then, the fundamental distinction between finiteness and infinity emanating from it. On the contrary, logicism abandons the distinction between infinity and finiteness[88] (being relevant only to the extensional viewpoint of set theory or arithmetic rather than to the proper viewpoint of propositional logic), heralding the unification of it with the intensional viewpoint of logicism and thus, the insignificance of the distinction between finiteness and infinity in the final analysis. Husserl's "epoché" now transferred "to infinity" (from its original meaning and philosophical sense inherent for phenomenology to be "to reality"[89]) can synthesize the attitude of logicism to the troubles about the foundations of mathematics.

Then and as a result, the multiplicative axiom and the axiom of choice cannot be absolutely identified with each other in the following sense. The multiplicative axiom is not able to distinguish finiteness from infinity initially and intentionally due to the leading idea of logicism unlike the axiom of choice literally referring only to infinite sets and thus distinguishing them in definition. That maybe scholastic (at first glance only) difference can be demonstrated by their opposite directions even in relation to the well-ordering theorem, namely:

Zermelo inferred (even in two different ways and papers: *Zermelo 1904; 1907*) the well-ordering theorem from the axiom of choice (in fact, newly introduced by him just for that purpose and seeming to him to be obvious unlike the well-ordering theorem itself). On the contrary, Whitehead and Russell deduced quite elementarily the multiplicative axiom from the well-ordering theorem. This means: the (eventual) equivalence of the axiom and choice and the well-ordering theorem or "theorem" (i.e. with or without quotation marks) relies on the (also eventual) equivalence of the multiplicative axiom and the axiom of choice acquitted by default by common sense, but rather doubtfully or ambiguous as this is shown above. Indeed, one can trace the same mismatch in detail and explicitly in relation to the well-ordering theorem in order to interpret it as their mutual duality, complementarity or even idempotency, as a result of which the one is a corollary from the well-ordering "theorem", but the other implies it (i.e. as a theorem without quotation marks).

Meaning that objective, the eventual interpretation of the well-ordering theorem or "theorem" as a true paradox as this is demonstrated above or in another paper (Penchev 2022 October 21) to the Gödel insoluble statement mediating between arithmetic and set theory in order to generate the Gödel dichotomy about their relation ("either incompleteness or contradiction"). The well-ordering (")theorem(") as a true paradox in the context at issue might mean the following: its finite, arithmetic interpretation implies its infinite, set-theoretical interpretation as well as *vice versa*: its infinite, set-theoretical interpretation implies its finite, arithmetic interpretation. Both can be anyhow consistent to each other, but only from the viewpoint of logicism and the multiplicative

[88] On the contrary, their distinction implies a "critical view of logic" (Parsons 2015) as well as an alternative viewpoint to arithmetic (Parsons 2013).
[89] One can link Husserl's original epoché to reality to the newly introduced epoché to infinity also e.g., by means of the "concept of evidence" (Snyder 1981).

axiom; on the contrary, they constitute a true paradox from the alternative viewpoint of set theory and the axiom of choice.

Thus, the conclusion about the relation of the axiom of choice and the multiplicative axiom turns out to be rather extraordinary: they are indeed equivalent, but only from the viewpoint of multiplicative axiom for the missing distinction of finiteness and infinity; on the contrary, they are dual or complementary from the viewpoint of the axiom of choice, because of the explicit distinction of infinity from finiteness. Thus, stating their equivalence (as usual), the viewpoint of logicism is granted in fact though implicitly[90].

Just the same ambiguity can be traced in the "nonstandard bijection" really being a bijection only from the viewpoint of logicism (or "intensionally", speaking loosely, i.e. if one has not defined before that what means the "bijection of two propositions"), but not being any bijection from the viewpoint whether set theory or arithmetic (or "extensionally", i.e. properly as far as the usual "bijection of sets" is inherently extensional). Then, the main idea of Hilbert arithmetic in a narrow sense for doubling Peano arithmetic by a dual and anti-isometric counterpart can be interpreted as a formal and rigorous translation of the viewpoint of logicism into the inherently extensional viewpoint of both arithmetic and set theory though opposed to each other by the axiom of induction versus that of infinity accordingly.

The axioms of set theory can be divided in a few groups according to the viewpoint of the present paper: (1) "Boolean axioms" defining algebraic operations for Boolean algebra; (2) "axioms for infinity": the axiom of infinity versus the axiom of induction in arithmetic as well as axioms for the Cantorian hierarchy of infinities; (3) the axiom of choice[91]. The effects of (2) and (3) can be interpreted as opposite to each other. The hierarchy of infinities assists to differentiate many kinds of infinities from each other according to their cardinal numbers: so-called "alefs". On

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[90] Sometimes, the "problem of infinity" is alleged in relation to logicism (e.g. Landini 2011; also Landini 2020). In fact, logicism does not suggest the distinction of infinity versus infinity. It abstains from that distinction or from the extensional viewpoint in general, at least in the interpretation of the present paper.

[91] That statement is quite not obvious and needs a relevant detailed proof postponed for a future paper about Russell's paradox in Hilbert arithmetic and intended to be published in the next year (2023). Speaking loosely, set theory establishes two specific concepts relevant to extensionality: sets and elements. Then, all "Boolean axioms" postulate the properties of sets and elements to propositional logic so that "set" and "proposition" are identifiable. If one speaks of the axioms of ZFC set theory (only for certainty since it is equivalent to the other standard axiomatics of set theory), both rest groups of axioms, (2) and (3) above, introduce specific properties of extensionality without any analogue in propositional logic and refers correspondingly to infinity (and to the hierarchy of infinities) and well-ordering: i.e. to arithmetic though indirectly rather than to propositional logic. Arithmetic is thoroughly finite due to the axiom of induction and thus, a true subclass of set theory admitting infinity and its hierarchies as well as inaccessible cardinal numbers (whether uncountable or countable) corresponding to the Gödel incompleteness statement; e.g. Corazza (2010) relates the axiom of infinity to them; Belyakin and Ganov (2002) mean "large cardinal numbers" in relation to intensionality. The axiom of choice supplies a countable well-ordering of any set (correspondingly finite or infinite), and arithmetic is a mathematical theory of all finite well-orderings. Thus, if "Boolean axioms" relate set theory to propositional logic (both being the same structure of Boolean algebra), all the rest, i.e. "non-Boolean axioms" refer to the relation of set theory to arithmetic. That is: the Boolean axioms elucidate that "set" and "proposition" are the same, after which the non-Boolean axioms specify what "elements" mean, implicitly relating set theory to arithmetic as a theory of all finite sets.

the contrary, the axiom of choice is able to reduce all infinities regardless of their cardinal numbers to a single one, countability in the sense of the Löwenheim - Skolem theorem or respectively, Skolem's "relativity of the concept of set".

Furthermore, the axiom of choice is able to generate one or two well-orderings for any infinite set unlike the single one well-ordering relevant to a finite set and utilized by the concept of natural numbers or the axiom of induction also excluding any alternative well-ordering featuring the well-ordering of any infinite set, in virtue of the axiom of choice[92]. The identification (whether absolute or partial) of the axiom of choice and the multiplicative axiom as above allows for the distinction of two well-orderings versus a single one as relevant to any infinite set, for example representable also by the non-standard bijection.

One can immediately visualize how the necessary utilization of the axiom of choice for any infinite set to be well-ordered generates one or two well-orderings equally well. The standard consideration relevant to a single one is the following. Some element of the infinite set at issue is chosen in virtue of the axiom of choice. The procedure is repeated to the set-theoretical complement of the chosen element or elements as a subset of the initial infinite set to the later itself iteratively. The procedure can be continued *ad lib*. Then, a countable set of elements (the initial infinite set, which is to be well-ordered) is mapped by a bijection into another countable set of choices. Finally, the procedure of well-ordering has to finish in virtue of that bijection.

However, one can admit the following alternative consideration: one doubles the afore-described procedure also in relation to the *unchosen* alternative, which can be identified with the complement of the chosen alternative to the initial set and it is continued analogically and iteratively *ad lib*. The process of doubling can be visualized as follows:

1	C (1)
1, 2	C (1,2)
1, 2, 3	C (1,2,3)
…	…
1, 2, 3, …, n	C (1,2,3, …, n)
…	…

So, the axiom of choice is able also to generate two isomorphic well-orderings, or more precisely: two dual anti-isometric Peano arithmetics. Any finite set can be well-ordered only so that a single one ordinal number corresponds unambiguously to it. On the contrary, the well-ordering by virtue of the axiom of choice is able to offer also two different ordinal numbers, each of which refers either to the chosen alternatives or to the unchosen alternatives. Nonetheless those two dual ordinal numbers relevant to any infinite set can be identified to be the same in the final analysis. So, the distinction between only a single one ordinal number versus two dual ones can furthermore to differentiate finiteness form infinity (speaking loosely) just any finite set, from any infinite set (speaking precisely)

The so-described construction by the axiom of choice is homomorphic to the previous one considering arithmetic to be the one "half of propositional logic", however without involving the

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92 Streicher (1992) investigates the relation of the axiom of choice and induction in a specific context.

axiom of choice since it is relevant to set theory rather than to propositional logic. The homomorphism of both corresponds to Boolean algebra underlying both.

V WHY QUANTUM MECHANICS DOES NOT NEED ANY "QUANTUM LOGIC"

As this is well known, the conclusions of quantum mechanics are so extraordinary that a relevant change of scientific thought seemed to be necessary initially, respectively a special quantum logic (or logics) corresponding to quantum revolution. When quantum mechanics appeared in the beginning of the 20th century, many non-classical logics, or logics of certain subjects, directed to describe such a way of thinking that fits very well only to a single scientific area, thrived. The basis (as in the case of non-Euclidean geometry) was the same deductive and axiomatic approach after complementing or modifying one or more axioms of classical logic in order to agree with the domain meant at issue.

According to classical logic, any mathematical theory shares the same list of the axioms of propositional logic, to which are added independent axioms describing its specific subject and transforming the corresponding complete tuple of axioms into a first-order (or eventually higher order) logic, properly. One can think of the distinction between the axioms of propositional logic and those of the first-order logic at issue to be conventional therefore admitting to be moved so that axioms specific for a certain mathematical theory (respectively, a first-order logic) to be included in the core, to which only those of propositional logic belonged initially (i.e., before that).

Furthermore, the newly added axioms to the core list can be absolutely or partly equivalent to negations or modifications of certain axioms of propositional logic. Since Boolean algebra is a lattice, different generalized lattices can correspond to the various lists of core axioms featuring one or another non-classical logic relevant to a corresponding area of cognition, respectively a mathematical theory interpreted to be a model of the area at issue[93].

The same mechanism of occurrence is shared by the class of quantum logics[94] and thus it can be traced back into their origin[95]. All of them can be featured by the propositions about conjugate quantities obeying the rules of complementarity or respectively and equivalently, the non-

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[93] For example, Mittelstaedt (2012; 2004); quantum logic and classical logic admit a unified interpretation (Tokuo 2014; Malhas 1987), or the "classical foundations of quantum logic" to be researched (Garola 1991).

[94] For example, Beltrametti, Mączyński (1995); Lock, Hardegree (1985); Birkhoff, Neumann (1936); or as a problem, by Gardner (1971).

[95] Many papers relate quantum logic(s) to different specific subjects (e.g. Matvejchuk, Vladova 2019; Baltag, Smets 2011; 2008; Pykacz 2010; 1998; 1992; Harding 2009; Engesser, Gabbay 2002; Bigaj 2001; Pulmanová 1998; 1996; 1994; 1983; 1981; 1977; 1973; Dalla Chiara 1995; Friedman 1994; Bell 1986; 1985; Brabec, Pták 1982; Dvurečenskij 1992; Grib, Zapatrin 1992; Finkelstein 1992; Cohen 1989; Cohen, Svetlichny 1987; Gibbins 1987; Kallus, Trnková 1987; Bunce, Wright 1985; Cook, Rüttimann 1985; Hughes 1985; Abbati, Manià 1984; Gudder 1982; Cook 1978; Cirelli, Cotta-Ramusino, Novati 1974; Cirelli, Cotta-Ramusino 1973) or to philosophy (Ashcroft 2010; Barnum 2003; Resconi, Klir, Pessa 1999; Pitowsky 1989; Mittelstaedt 1986; Bell, Hallett 1982; Hardegree 1977; Nilson 1977; Putnam 1974), especially in the second half of the 20th century where its peak is. However, all quantum logics are rather "exercises" and variations within logic, not contributing anything in quantum mechanics, physics, or science.

commutativity of Hermitian operators in the separable complex Hilbert space after the definition of 'quantity' in quantum mechanics to be that operator[96]. Then, that special class of propositions is regulated by relevant axioms[97]. Quantum logic can be furthermore extended to quantum information (e.g., Hyttinen, Paolini, Väänänen 2015; Barnum 2003; Santos 2003; Pulmanova 2002), which can be also "analyzed by ordinary mathematical logic" (Nisticò 2014).

Howcwer, the approach of quantum logic has not been accepted by the physicists in the area of quantum mechanics[98]. They continued and continue to use classical logic[99]: i.e. propositional logic to a special mathematical theory referring to operators in Hilbert space as a model of the physical and experimental discipline of quantum mechanics[100]. Nonetheless, the theorems about the absence of hidden variables are deduced though being overly surprising to classical science or classical logic.

They can be also interpreted as the redundancy of quantum logic thus cut by Occam's razor. Indeed, the propositions about conjugate quantities cannot contain any additional information in comparison with the propositions obeying classical logic since that eventual additional information can be related to some "hidden variables", which cannot exist according to the theorems cited above and hence implying the equivalence of quantum and classical logic or the redundancy of the former.

The same redundancy of quantum logic to classical propositional logic can be simply illustrated by means of Hilbert arithmetic after using the two dual Peano arithmetics[101] for enumerating all pairs of conjugate quantities. Then, the equivalence of quantum logic and classical propositional logic can be reduced to granting the nonstandard bijection (and consequently, to the Gödel incompleteness statement in the final analysis, however only in its interpretation above, by means of Hilbert arithmetic). Accordingly, the redundancy of quantum logic corresponds to the completeness theorem after proving its equivalence to the compactness theorem.

So, quantum logic (respectively, the class of quantum logics) turns out to be an extremal, but very instructive case of non-classical logic generating a series of ambiguous interpretations and linked problems. It can be realized as a logic of arithmetic after emphasizing that the theorems about the absence of hidden variables imply the identification of it with propositional logic in the

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96 For example, in Ross (1974), Lahti (1980); also, Torza (2020) or Tomé, Gudder (1990) as well as in the context of symmetries of quantum logics (as in: *Trnková 1989; 1984*) or in Schindler (1992), by the "existence property".

97 For example, in Zecca (1981); Gudder, Michel (1979); Gudder (1969).

98 For example, Foulis, Randall 1974; Strauss (1973) considers the problem about "logics for quantum mechanics"; Renauld and Joachim (2011) discuss the application of classical Boolean logic to quantum systems; Dieks (2014) compares classical and quantum physics in relation to identity (being fundamental for logic), distinguishability and indistinguishability, and Faggian, Sambin (1998), to cut-elimination.

99 This can be demonstrated rather elegantly following Calabrese (2005) and the interpretation of quantum logic by "Boolean fractions": then they may be re-enumerated into Boolean "integers" relevant to classical propositional logic.

100 Described (e.g.) by Hooft (2012); Deutsch, Ekert, Lupacchini (2000); or Zapatrin (1994). El Nascie (2007) links infinity and the physical theory of E-infinity by Hilbert space. The logic of quantum mechanics derives from "classical general relativity" according to Haddley (1997).

101 For example, Dacey (1990) discusses "arithmetic tools for quantum logic".

final analysis. Indeed, arithmetic as any other mathematical theory is a first-order (or eventually higher-order) logic, consequently granting all axioms of propositional logic in advance. Then, if quantum logic is accepted to be the non-classical logic of a special area, namely arithmetic, it possesses the particular (or even maybe unique) property to coincide with propositional logic so that arithmetic whether as a non-classical logic called quantum logic or as a first-order logic is the same.

Another viewpoint is that quantum logic is to be interpreted as the relation of propositional logic to its "half" as arithmetic (following the construction above). For example, if propositional logic and quantum logic coincide according to the theorems about the absence of hidden variables, this implies a whole and its half to coincide, which is quite natural if both are actually infinite sets being also absolutely consistent with the identification of set theory and propositional logic as Boolean algebra[102].

A third interpretation is possible as well, furthermore being the most relevant one in the present context about logic as ontology. Quantum logic can be realized as the class of all non-classical logics of any extensional "something" (whatever it be[103]) analogically to set theory if it is granted to be a class of all first-order logics. Then, the relation of propositional logic to quantum logic can be proved to be isomorphic to that of it to set theory[104] and in the final analysis, again identifying quantum logic and propositional logic, but now by the mediation of set theory equated to both in advance.

What is to be added to propositional logic (respectively, to the list of its axioms) is one more bit of information consisting of the following two oppositions: (1) intensionality for propositional logic versus extensionality for quantum logic; (2) finite extensionality for arithmetic versus infinite extensionality for set theory (or the corresponding pair of two dual Peano arithmetics according to Hilbert arithmetic). So, quantum logic can be thought to be propositional logic, though being inherently intensional, now related to extensionality: or speaking loosely, the equivalent extensional counterpart of propositional logic.

On the other hand, ontology being the logic of the world can be identified as the extensional counterpart of propositional logic being intensional by itself, and thus, with quantum logic in the final analysis. The world needs conjugate quantities, noncommutative Hermitian operators in quantum mechanics to be described. Quantum logic corresponds to that description as well as to propositional logic linking the former to the latter by the nonstandard bijection.

So, quantum mechanics does not need any quantum logic since classical propositional logic only reinterpreted ontologically is absolutely sufficient for it also in virtue of their fundamental coincidence as above.

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[102] The problem about "completeness of quantum logic" (Stachow 1976) can be considered in the same context.

[103] For example, meant by a physical theory, i.e. being in the framework of physics (as in: Svetlichny 1992).

[104] Stout (1979) demonstrates a "quantum-logic-valued model of set theory".

VI HILBERT'S "EPSILON CALCULUS" AFTER HILBERT ARITHMETIC IN A WIDE SENSE AND LOGICISM

The epsilon operator, "ε", distinguishing between propositional logic and any first-order logic (and thus between propositional logic and set theory as the class of all first-order logics) unifying both quantifiers ("∀" and "∃") is crucial in Hilbert's idea for a mathematical theory[105] able to prove that mathematics is complete (e.g. Slater 1994; 1991). In fact, it does not add any essentially new concepts in mathematics, its foundations or philosophy, however emphasizes both quantifiers reducible to the single operator at issue as necessary and sufficient to justify completeness, or particularly, to overcome the Gödel incompleteness statement[106].

Hilbert's ε-operator serves only to notate the well-known mathematical idea of "pure" or non-constructive existence provable[107], for example, by *reductio ad absurdum*. So, it being related to the problem about completeness hints or suggests that eventual incompleteness is due only to the restriction of constructiveness[108]: accordingly, if one removes it by admitting the option for mathematical entities to exist regardless of the eventual fact that any constructive proof is impossible, the completeness of mathematics can be proved. One can immediately assure that the Gödel incompleteness statement is not valid to non-constructive proofs about pure existence since the involvement of arithmetic (for example in virtue of the Gödel enumeration) implies constructiveness.

An illustration of the distinction between propositional logic and first-order logic (even in the case of set theory as the class of all first-order logics being equivalent to propositional logic in the sense of the completeness theorem) as inherent and definitive for the ε-operator can be the "paradox of material implication" especially after its utilization to demonstrate the eventual non-constructiveness of the Gödel incompleteness statement as in *Part I* of the present paper. Indeed, the ostensible "paradox of material implication" is due to the incorrect identification of implication in propositional logic (i.e., without any additional axioms and without quantifiers) and implication in any first-order logic, sometimes called "material implication" for being distinguishable (i.e. after additional axioms in the first-order logic or admitting quantifiers).

The paradox consists in the possible absurdity of true implications according to propositional logic where the link between their interpretations in terms of first-order logic is meaningless. If one involves ε-operator, which is quite relevant after interpreting the paradox of material implication by the relation of propositional logic and first-order logic, the paradox can be resolved admitting in an absolutely non-constructive way that there exists some meaningful premise to

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105 Gauthier (1994) discusses Hilbert's "internal logic of mathematics".

106 For example, as in Zach (2003).

107 For example, Mints (2008) demonstrates cut-elimination for epsilon calculus Moser and Zach (2006) links it with Herbrand complexity. Koslow (2019) considers the "modality and non-extensionality of the quantifiers."

108 Constructiveness is always consistent with finiteness, but not always with infinity (e.g., Rowbottom 1971).

justify the conclusion also in the framework of the first-order logic rather than only abstractly, in propositional logic[109].

As to logicism, Hilbert's epsilon calculus reduces the problem of completeness[110] to the relation of propositional logic and all first-order logics featured by both quantifiers. After set theory has been identified as a special mathematical theory about the class of all first logics and the completeness theorem has been proved, the pathway to Hilbert's epsilon calculus is pioneered: its contribution consists only in the definitive distinction of propositional logic to set theory or to any first-order logic by means of ε-operator.

Speaking loosely, propositional logic complemented by ε-operator can be interpreted as a first-order logic or as the class of all first-order logics, i.e as set theory. However, that ε-operator vanishes (or respectively cannot be defined) in the framework of the completeness theorem (1930) since the distinction between constructiveness and non-constructiveness or the ε-operator relying on their relation cannot be formulated. Utilizing again the biblical metaphor about the "expulsion from Paradise", one may say that there is no ε-operator "in Eden"; it can be defined only after "expelling" from there.

If one interprets the ε-operator in terms of Hilbert arithmetic, it should be related to the complementary or dual Peano arithmetic (notated for example as PA2) if all constructive proofs are referred to its counterpart (naturally notated to be PA1). PM2 admits only uncertain statements about existence (which can be identified to be "pure existence" as in mathematics) just due to its duality or complementarity to PM1. For example, PM2 can be related also to the Gödel insoluble statements thus equating them to the area of pure existence or to the domain where the ε-operator can be defined by mediation of PA2.

One can reveal some ostensible inconsistency between the Gödel insoluble statements and the area of statements about pure existence of mathematics, since the latter are not insoluble. In fact, the former, the Gödel insoluble statements are to be called more precisely "constructive insoluble statements" since their insolubility relies only on their Gödel numbers being inherently constructive. On the contrary, the statements for which neither constructive nor pure existence is provable are false rather than belonging to the area of Gödel insoluble statements.

The correspondence of ε-operator and the dual counterpart of Peano arithmetic can be made clearer by means of Hilbert arithmetic in a *wide sense*, which includes both dual qubit Hilbert spaces so that the dual counterpart of Peano arithmetic can be identified to be the one qubit Hilbert space. Then, the pure existence suggested by ε-operator can be identified with a probability (density) distribution being uncertain in comparison with any constructive unambiguous procedure[111]. That probability distribution corresponds to a wave function and then to one or more qubits according to the mathematical formalism of quantum mechanics and information[112].

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[109] The quantifiers can be related to intensionality (e.g., Dalrymple, Lamping, Pereira, Saraswat 1997).

[110] Epsilon calculus is to be thought in the framework of Hilbert's program (e.g. Franks 2009; Detlefsen 1986).

[111] Bugajski (1978) compares the logics of classical and quantum mechanics to probability.

[112] For example, Horgan (2000) investigates the "intensionality of probability" though in a specific context.

The proper idea of the ε-operator being uncertain in definition is now translated more or less successfully into the language of possibility or even into the rigorous mathematical language of probability acquiring partly the certainty of some probability distribution distinguishable from any other referring to probabilities of the same option. So, Hilbert's proper idea of ε-operator can be now defined as the class of all possible probability distribution or respectively, as an "empty qubit" being exactly so uncertain as the ε-operator. Then, an "empty qubit" can be interpreted as equivalent to the set of all natural numbers relevant to the dual counterpart of Peano arithmetic therefore restricting the description only to Hilbert arithmetic in a narrow sense in the final analysis, but necessarily involving the concept of set.

Returning to logicism, the ε-operator referring only to both quantifiers can be equivalently related to the relation of propositional logic and first-order logic thus remaining only in the framework of logicism and thus of the completeness theorem not needing even the compactness theorem. Indeed, the latter means rather the relation of the two dual Peano arithmetics of Hilbert arithmetic; and the ε-operator: only relevant properties of the dual counterpart of Peano arithmetic rather than any relations of them.

VII INTERIM CONCLUSION: HILBERT MATHEMATICS AS THE LOGICIST FOUNDATIONS OF MATHEMATICS

The next *Part III* of the paper will consider Hilbert mathematics opposed to Gödel mathematics[113] in detail. It relies on Hilbert arithmetic in both narrow and wide senses and corresponds to a kind of Pythagorean philosophy called "quantum neo-Pythagoreanism", according to which the essence of the world is mathematical and it exists by virtue of mathematical necessity.

Common sense's viewpoint that mathematics can only create models of the world, which more or less correspond to the world "by itself" is only a meta-metamathematical axiom, the proper mathematical counterpart of which (also an axiom) is the Gödel incompleteness statement and which can be accepted or not equally well just as the Fifth postulate of Euclid can be accepted or not. The previous *Part I* of the paper contains enough arguments and tenets demonstrating that it is an axiom rather than a theorem regardless of Gödel's proof or the corresponding common opinion.

However, a modified Pythagorean philosophy, in which the fundamental and even sacral arithmetic of original Pythagoreanism is replaced by logic, therefore granting for the world to be ontology, is quite usual for philosophy since Aristotle's age. Unlike geometry or arithmetic, logic is related to language[114] or human thought rather than, to mathematics during more than two millennia: only since the second half of the 19th century, propositional logic has been identified as a mathematical discipline after Boolean logic and algebra. Then, ontology as a class of

[113] One can speak of "Gödel mathematics" also in virtue of his viewpoint to philosophy of mathematics usually qualified to be Platonist (e.g. Fuchino 2012; Tieszen 2011; 2006; 1998; Taotao 2011; Odifreddi 2011; Pleitz 2010; Tymoczko 1998; Headley 1997; Sapojnikoff, Sapojnikoff 1973; Silvers 1966).

[114] For example, extensionality is frequently considered as a linguistic concept (or both linguistic and logical) as e.g. by Humberstone (1986) or by Lambert (2005; 1974).

philosophical doctrines can be understood as a form of Pythagoreanism: where arithmetic as a mathematical structure is replaced by another mathematical structure, that of propositional logic after it has been identified as Boolean algebra.

Russell's logicism, being a school for the foundations of mathematics after Cantor's set theory and its paradoxes, can be also understood as a form of Pythagoreanism in an analogical way though in a quite narrow sense, relevant to philosophy of mathematics regardless of being only an application of the fundamental conception of ontology. Gödel's two papers (1930; 1931) can be interpreted as advocating logicism, but also in a wide sense, according to which logicism can be understood ontologically, i.e. as a form of Pythagoreanism, as the present *Part II* of the paper demonstrates.

However, logicism or ontology interpreted as relevant to Pythagoreanism are justified whether only intensionally or by Husserl's "epoché" to reality correspondingly, thus being a too restricted version of Pythagoreanism or its original ambitions. On the contrary, both quantum neo-Pythagoreanism and Hilbert mathematics claim to expand Pythagoreanism according to its ancient ambitions and initial intentions: also, to extensionality being inherent for arithmetic and set theory or to reality by itself revealing its mathematical basis.

The final *Part III* intends to investigate Hilbert mathematics both fundamentally and philosophically, on the one hand, and as philosophy of mathematics, on the other hand. Furthermore, the viewpoint of Hilbert arithmetic allows for a series of the most fundamental problems to be resolved: for example, the "four-color theorem"; Poincaré's conjecture proved by G. Perelman, proclaimed by CMI to be one of the Seven Problems of the Third Millennium; Fermat's last theorem proved by A. Wiles.

The following hypothesis can generalize the approach only illustrated by those three fundamental mathematical problems: many of the most fundamental unsolved mathematical problems nowadays are to refer to the foundations of mathematics, to its philosophy and relation to the world, and even to the "first philosophy" itself.

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