Proof rules for the dialogical logic N*

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Abstract

We outline the beginnings of a proof-theory for the dialogical logic N introduced in [1].

1 Introduction

Dialogical logic was introduced by Lorenzen in the 1950s and 1960s as an alternative semantics for intuitionistic logic [5, 6] based on the existence of *winning strategies* for certain two-person games. The dialogical approach has been extended beyond intuitionistic logic to give new semantics for classical logic, modal logic, free logic, connexive logic, relevance logic, and others [3, 7]. However, with one exception, all work on dialogical logic to date has been done with respect to logics which are independently defined, that is, where a semantics or proof theory for the logic is already known. The exception [1] uses dialogues not to provide semantics for a known logic but to generate a wholly new sub-classical logic, N. Because the only known semantics for N are dialogical, a number of interesting open problems arise for it, such as whether an axiomatization or proof theory can be given for it, and whether a more standard, non-dialogical semantics can be given. Our focus in this note is the former question; we make preliminary investigations into the proof theory for N.

2 Dialogical logic

The basis of the dialogical approach to logic is finitary open two-person zero-sum games between the players Proponent P and Opponent O [4]. A formula φ is S-valid iff Proponent has a winning strategy for φ, according to some designated set S of rules governing allowed dialogues. The dialogical rules are divided into two types, structural and particle. The particle rules relate solely to the logical connectives; they explain how formulas can be attacked and defended by the two players based on the structure of the formulas. Structural rules restrict the possible moves of the players at any given round of the dialogue [2]. In addition to propositional formulas, there are the three so-called *symbolic attack* expressions, ?, $\wedge_L$, and $\wedge_R$.

The standard particle rules for a basic propositional language are given in Table 1; they specify which attacks and defenses can be used against formulas of various types. Note that there is no way to defend against an attack against a negation; the only appropriate "defense" against an attack on a negation $\neg\varphi$ is to continue the game with the new information φ. We introduce the structural rules under investigation in the next section; first we continue with some general definitions about dialogues.

*Both authors were funded by the FCT/NWO/DFG project "Dialogical Foundations of Semantics" (DiFoS) in the ESF EuroCoRes programme LogICCC (FCT LogICCC/0001/2007; LogICCC-FP004; DN 231-80-002; CN 2008/08314/GW).

Assertion	**Attack**	**Response**
$\varphi \wedge \psi$	$\wedge_L$	φ
	$\wedge_R$	ψ
$\varphi \vee \psi$	?	φ or ψ
$\varphi \to \psi$	φ	ψ
$\neg\varphi$	φ	—

Table 1: Particle rules for dialogue games

Definition 2.1. Given a set S of structural rules, an S-*dialogue* for a formula φ is a dialogue commencing with φ that adheres to the rules of S. P *wins* an S-dialogue if there is a round where P has moved and there is no move that O an legally make.

We can represent the development of dialogues as trees. The S-*dialogue tree* $T_{S,\varphi}$ for a formula φ is a rooted tree such that every branch of $T_{S,\varphi}$ is an S-dialogue for φ, and every S-dialogue for φ occurs as a branch in $T_{S,\varphi}$. We then identify a subset of branches of $T_{S,\varphi}$ which we designate as winning.

Definition 2.2. An S-*winning strategy* s for P for φ is a rooted subtree of $T_{S,\varphi}$ satisfying:

1. The root of s is the root of $T_{S,\varphi}$;
2. Every branch of s is an S-dialogue won by P;
3. If k is odd and a is a depth-k node of s, then a has exactly one child;
4. If k is even and a is a depth-k node of s, then all of a's children in $T_{S,\varphi}$ are present.

That is, a winning strategy for P is a kind of function saying how P can win given any move by O.

3 The logic N

The logic N is the set of formulas for which P has a winning strategy using the particle rules specified previously and the following structural rules (following Felscher [2]):

(D10) P may assert an atomic formula only after it has been asserted by O before.

(D13) A P-assertion may be attacked at most once.

That this set of formulas is closed under modus ponens, that is, that it is a logic, is Theorem 5 of [1].

N is a sub-classical logic; it does not, for example, validate Peirce's law $(((p \to q) \to p) \to p)$. In fact, the following characterization of valid implications in N can be given:

Theorem 3.1 ([1], Thm. 3)**.** *Every* N*-valid implication $\varphi \to \psi$ satisfies one of the following conditions: (1) φ is atomic; (2) φ is negated; or (3) ψ is* N*-valid.*

N is neither sub- nor super-intuitionistic: It validates the Law of Excluded Middle, so it is not subintuitionistic; and it does not validate one of the intuitionistically valid forms of De Morgan's Laws $((\neg p \vee \neg q) \to \neg(p \wedge q))$, so it is not superintuitionistic. Further properties of N include:

$$\frac{\vdash_{\mathsf{N}} \varphi \to \psi}{\vdash_{\mathsf{N}} \neg\psi \to \neg\varphi}\ \text{Contraposition} \qquad \frac{\vdash_{\mathsf{N}} \varphi \qquad \vdash_{\mathsf{N}} \varphi \to \psi}{\vdash_{\mathsf{N}} \psi}\ \text{Modus ponens} \qquad \frac{\vdash_{\mathsf{N}} \psi}{\vdash_{\mathsf{N}} \varphi \to \psi}\ \text{Weakening}$$

$$\frac{\vdash_{\mathsf{N}} \varphi}{\vdash_{\mathsf{N}} \neg\neg\varphi}\ \text{DNI} \qquad \frac{\vdash_{\mathsf{N}} \neg\neg\varphi}{\vdash_{\mathsf{N}} \varphi}\ \text{DNE} \qquad \frac{\vdash_{\mathsf{N}} \varphi \to (\psi \to \theta)}{\vdash_{\mathsf{N}} \psi \to (\varphi \to \theta)}\ \text{Exchange}$$

Figure 1: Sound proof rules of N

Lemma 3.2 ([1], Lem. 1)**.** *If* $\vDash_{\mathsf{N}} \psi$*, then* $\vDash_{\mathsf{N}} \varphi \to \psi$*, for all formulas* φ*.*

Lemma 3.3. $\vDash_{\mathsf{N}} \varphi$ *iff* $\vDash_{\mathsf{N}} \neg\neg\varphi$.

Proof. ($\Leftarrow$) This follows from the proof of [1, Thm. 5].

($\Rightarrow$) This is an immediate corollary of [1, Thm. 4], since $\vDash_{\mathsf{N}} \varphi \to \neg\neg\varphi$. □

Unlike other dialogical logics which already have independent semantic and proof-theoretic justifications, N is currently characterizable only by dialogical means. In the next section we sketch the beginnings of an axiomatization and proof-theory for N.

4 Proof rules and axioms for N

4.1 Axioms

A list of some N-valid formulas is given in [1, Table 3]; this list includes the Law of Excluded Middle, Weak Excluded Middle, Dummett's formula, double negation introduction and elimination, the K formula, Conditional Excluded Middle, and two of the four implications of DeMorgan's laws. Because none of them contain as subformulas any N validity (and hence cannot be reduced to other valid formulas), all of these are to be considered candidates for axioms of N.

4.2 Proof rules

Figure 1 lists some rules of inference that are known to be sound for N. The soundness of modus ponens follows from the positive solution to the composition problem for N [1, §4]. Lemmas 3.2 and 3.3 express the soundness of weakening and double negation introduction and elimination.

To show that contraposition is sound, consider the partial dialogue trees given in Tables 2 and 3; the first is the beginning of the dialogue tree $T_{\mathrm{N},\varphi\to\psi}$ and the second is one branch of the dialogue tree for $\neg\psi \to \neg\varphi$. Note that the initial branch of the N-dialogue tree for $\neg\psi \to \neg\varphi$ already excludes a possibility: At step 2, P could have attacked O's assertion of $\neg\psi$, rather than defending. O's moves, however, are forced. Nevertheless, this branch can be extended to a winning strategy by taking the winning strategy for $\varphi \to \psi$, minus the first two steps displayed in Table 2, and increasing the move labels by 4. Such a grafting does not fail to account for possible moves of O beyond step 5 because O cannot attack or defend any moves in the initial 4-step sequence of Table 3 .

0	P	$\varphi \to \psi$	*(initial move)*
1	O	φ	[A,0]

Table 2: Initial segment of a winning strategy for $\varphi \to \psi$

To prove the soundness of the exchange rule, note that the characterization theorem for N-valid implications allows us to dispense with one possibility immediately: If θ alone is N-valid, then desired conclusion follows by two applications of weakening. If θ alone is not N-valid but $\psi \to \theta$ is, then after the first four moves of the dialogue tree for $\psi \to (\varphi \to \theta)$ (given in Table 4[1]), P can ignore the information φ of move 3 and use the winning strategy for $\psi \to \theta$. The most interesting case is when neither θ nor $\psi \to \theta$ are N-valid. We can recover a winning strategy for $\psi \to (\varphi \to \theta)$ by comparing Table 4 with the information we have thanks to the assumption that $\vDash_{\mathsf{N}} \varphi \to (\psi \to \theta)$. The opening of the N-dialogue game for this formula is given in Table 5. Note that in Table 4, the formulas φ and ψ have both been asserted by O, but the order of their assertion is exchanged compared to Table 5. It is clear that P has a winning strategy for $\psi \to (\varphi \to \theta)$ for which the first four moves of Table 4 constitute an initial segment; simply permuting the use of the information φ and ψ in the winning strategy for $\varphi \to (\psi \to \theta)$.

Curiously, the following proof rule is *not* sound:

$$\frac{\vdash \varphi \to \psi}{\vdash (\gamma \to \varphi) \to (\gamma \to \psi)}$$

As a counterexample, take the N axiom $\neg\neg p \to p$. Let $\gamma := q$. The formula $(q \to \neg\neg p) \to (q \to p)$ is not N valid because the antecedent is neither atomic nor negated, and the consequent is not itself an N validity, and hence the implication fails to satisfy the characterization theorem.

4.3 Uniform substitution

It turns out that unrestricted uniform substitution is not admissible in N. For example, the substitution of $p \wedge p$ for p in any N-valid implication will not preserve N-validity, as the resulting implication will fail to meet any of the criteria in Theorem 3.1. Despite the general failure of uniform substitution, we have identified three special types of uniform substitutions that do preserve validity.

Lemma 4.1. *(1) Alphabetic renaming of atoms and (2) double negating atoms are validity preserving.*

[1] We are not considering the case where P attacks ψ at move 2, because then O could defend, but in N if O defends then P cannot win [1, Corollary 1] so such an opening of the game could not be the initial segment of a winning strategy.

0	P	$\neg\psi \to \neg\varphi$	*(initial move)*
1	O	$\neg\psi$	[A,0]
2	P	$\neg\varphi$	[D,1]
3	O	φ	[A,3]

Table 3: A branch of the dialogue tree $T_{\mathrm{N},\neg\psi\to\neg\varphi}$

0	P	$\psi \to (\varphi \to \theta)$	*(initial move)*
1	O	ψ	[A,0]
2	P	$\varphi \to \theta$	[D,1]
3	O	φ	[A,2]

Table 4: Initial segment of a winning strategy for $\psi \to (\varphi \to \theta)$

0	P	$\varphi \to (\psi \to \theta)$	*(initial move)*
1	O	φ	[A,0]
2	P	$\psi \to \theta$	[D,1]
3	O	ψ	[A,2]

Table 5: Initial segment of a winning strategy for $\varphi \to (\psi \to \theta)$

Proof. (1) A substitution s of atoms for atoms, suitably extended to formulas, dialogue games, and dialogue trees, clearly sends a dialogue tree $T_{\mathrm{N},\varphi}$ for a formula φ to an isomorphic dialogue tree $T_{\mathrm{N},s(\varphi)}$, since all the particle and structural rules are neutral with respect to atoms (that is, they do not mention any particular atoms).

(2) The atoms of a formula φ occur in an N-dialogue game in two ways, owing to the distinction between particle and structural rules. Since the particle rules break down a formula into its constituent subformulas, the first time an (occurrence of an) atom p occurs in an N-dialogue game will be via the particle rules; further uses of the (occurrence of the) atom are governed by the structural rules. We give an example: Transform the strategy in Table 6, where p_k first occurs at m, into the one in Table 7. It is clear that the only effect of this transformation is to delay the exposure of atoms.

We need to show that if there was a winning strategy for $\varphi(p_1, \ldots, p_k, \ldots, p_n)$, then there is a winning strategy for $\varphi(p_1, \ldots, \neg\neg p_k, \ldots, p_n)$. But if there is no winning strategy for $\varphi(p_1, \ldots, \neg\neg p_k, \ldots, p_n)$, then it is because O can stall by repeating attacks or defenses against either $\neg\neg p_k$ or $\neg p_k$. But O cannot attack P's assertions more than once, nor can O defend against attacks on negations. Thus, even though the game for $\varphi(p_1, \ldots, \neg\neg p_k, \ldots, p_n)$ is longer than the game for $\varphi(p_1, \ldots, p_k, \ldots, p_n)$, no new moves become available to O. □

The third type of uniform substitution that we have considered is the following:

Proposition 4.2. *Substitution of validities for atoms is validity preserving.*

We omit the proof due to space constraints.

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0	P	$\varphi(p_1, p_2, \ldots, p_n)$	*(initial move)*
$\vdots$	$\vdots$	$\vdots$	$\vdots$
m	P/O	p_k	[A/D,m']
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 6: An N-dialogue game for $\varphi(p_1, \ldots, p_n)$ where p_k is played

0	P	$\varphi(p_1, p_2, \ldots, \neg\neg p_k, \ldots, p_n)$	*(initial move)*
$\vdots$	$\vdots$	$\vdots$	$\vdots$
m	P/O	$\neg\neg p_k$	[A/D,m']
$m+1$	O/P	$\neg p_k$	[A,m]
$m+2$	P/O	p_k	[A,$m+1$]
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 7: An N-dialogue game for $\varphi(p_1, \ldots, \neg\neg p_k, \ldots, p_n)$, with two further attacks

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